Arrow of time in generalized quantum theory and its classical limit dynamics

Vadim V. Asadov[⋆1] and Oleg V. Kechkin[+⋆2]

[+]Institute of Nuclear Physics,
Lomonosov Moscow State University,
Vorob'jovy Gory, 119899 Moscow, Russia

[⋆]IIP–Neur OK,
Scientific park of MSU, Center for Informational Technologies–104,
Vorob'jovy Gory, 119899 Moscow, Russia

PACS No(s). : 05.30.-d, 05.70.-a.

Abstract

In this paper we have studied a generalized quantum theory and its consistent classical limit, which possess a well-defined arrow of time in their dynamics. The original quantum theory is defined as analytically dependent on complex time and specified by non-Hermitian Hamiltonian structure.

1 Generalized quantum theory

Irreversibility of evolution is a common feature of all real dynamical systems. This fact is reflected in the second law of thermodynamics, which states that entropy of any closed system can not decrease. Figuratively, one says about some 'arrow of time', which separates

[1]asadov@neurok.ru
[2]kechkin@depni.sinp.msu.ru

past and future in the absolute manner, and must be guaranteed by any realistic dynamical theory without fail [1]–[4].

Here it is important to stress, that thermodynamics itself is not a dynamical theory of the same type as, for example, classical or quantum mechanics. Rather, it provides some general theoretical framework for dynamical theories of this type, which pretend to the adequate description of evolving physical reality. Also, it can be said that thermodynamical principles must be realized on the base of the consistent dynamical theory of the fundamental type, and that thermodynamics imposes hard restrictions to corresponding theoretical constructions [5].

In this connection, one must take into account that all quantum and classical fundamental theories are conservative Hamiltonian systems [6]–[7]. All these systems are reversible in time, because one can interchange the initial and final data to obtain really possible result of their dynamical evolution. Thus, these theories, being the most natural concrete classes of the closed dynamical systems, can not be used for any consistent realization of the thermodynamic conception of the arrow of time. Actually, all attempts performed to 'average out' the reversible results of these theories to obtain the thermodynamic irreversible ones, contain hidden incorrect actions.

Also, impossibility to consider unidirectional evolution in framework of the classical mechanics can be understood using results of the Poincare theorem. Actually, it states, that one can decompose any classical motion to the set of the Poincare cycles, so this motion obtains transparently reversible character. The same situation takes place in the quantum mechanics: using the Hamiltonian proper basis one reduces quantum motion to the set of corresponding oscillations, which are reversible manifestly. As the result, the probabilities to find quantum system in the Hamiltonian eigenstates are constant quantities, and one can not relate any irreversible dynamics of the kinetic type to them.

In this point we would like to emphasize, that the set of conventional kinetic equations contain arrow of time by its definition. Such equations had been used extensively by Prigogine to study both the thermodynamic and synergetic processes [8]. The only conceptual problem of the all corresponding approaches is how to ground the kinetic method of realization of the time irreversible disciplines on the fundamental level in the really correct form. In the over words, one must show how to modify the fundamental theories of the classical and quantum mechanic types to achieve their consistency with conventional kinetic framework. In this paper we answer on this question by presentation of the corresponding generalizations of the both quantum and classical theories.

Namely, we consider some natural generalization of the conservative quantum theory, and show that its dynamics is actually irreversible. Our generalization is related to use of the both complex parameter of evolution (or 'complex time', [9]–[12]) and non-Hermitian Hamiltonian independent on this evolutionary parameter [13]–[16]. Also, we suppose analytic dependence of quantum state vectors on the evolutionary parameter and impose commutativity restriction on the Hamiltonian and the result of its Hermitian conjugation. We interpret real and imaginary parts of the complex time as ‘usual’ time and minus half of inverse absolute temperature, respectively. Then, we identify Hermitian part of the Hamiltonian with energy operator of the quantum system, whereas anti-Hermitian one we relate to operator of decay times for the energy eigenstates. Finally, we define thermodynamic regimes of evolution of the quantum system; this allows us to determine temperature functions for concrete thermodynamic processes considered in the paper. All these regimes are obtained by help of fixation of average values of the corresponding observable quantities and have natural and physically well-motivated sense.

The work is organized as follows. It consists of two parts, which describe the quantum and classical modified theories, both Hamiltonian ones and time-irreversible. The classical theory is derived as the corresponding limit of the quantum theory case; we study it under the same conditions as its fundamental quantum origin.

First of all, we present general scheme for modification of the quantum theory supplemented by method of fixation of the thermodynamic regime. After that, we study isothermal and adiabatic regimes in details and show, that both these types of evolution actually possess the arrow of time. It is proved, that effect of the irreversibility can be related to the non-increasing evolution of the average value of the anti-Hermitian part of the Hamiltonian of the system in these cases.

Then, we investigate dynamics of quantum subsystem of the generalized quantum system with two different values of the decay time parameter and study the entropy evolution of this subsystem. In particular, we show how one can guarantee existence of the well-defined arrow of time in this dynamics by some additional hipping of the decay properties of the vacuum state only.

Also, we study the non-Hermitian generalization of the quantum theory with discrete symmetry of the parity type and show, that physical evolution in this theory possesses well-defined arrow of time. The result of its evolution is related to transformation of the ‘left’ quantum world to the ‘right’ one (or vice versa) for the arbitrary thermodynamic regime. We have realized this general quantum scheme for massless Dirac particles to explain the problem of helicity symmetry breaking for the real neutrino in the natural framework of the modified quantum dynamics.

In the second part of the work we derive a modified Hamiltonian dynamics, which describes a classical regime of evolution of the quantum theory studied in the first part. We define the classical dynamics as such special limit of the quantum one, that is specified by resonance character of the probability density in the theory representation taken. In the ‘hard’ classical mode of the theory we put $\hbar \to 0$; in the ‘soft’ regime of its evolution we save the Planck constant as some free parameter.

Our starting point is the representation of the original quantum theory in terms of the quantum numbers, which contain the energy and decay operator indexes. We consider the field of ‘big values’ of the all quantum numbers and replace the originally discrete problem to the continuous one in the appropriate way. After that, we check out the consistency of the equations, which describe the corresponding ‘hard’ dynamics. Note, that this point is really nontrivial one, because this dynamics depend on the two evolutionary parameters: the physical time and the temperature. Thus, the ‘hard’ classical dynamics under discussion is, in fact, some special classical field theory, which is described by some system of partial differential equations. We show, that this classical dynamics possesses arrow of time like its time-irreversible quantum origin (in the isothermal and adiabatic regimes of evolution, at least).

Then, we establish a symplectic group of hidden symmetries for the leading term of the modified classical Hamiltonian equations for the ‘soft’ dynamics with $\hbar \neq 0$. Namely, we have shown, that in this case one deals with the symmetry group $Sp(2N,\, R)$ for the theory with N degrees of freedom. It is shown, that this symmetry corresponds to some subgroup of the group of conventional canonical transformations. The most interesting opportunity is related to fact that this hidden symmetry acts not only on the ‘usual’ Hamiltonian part of the equations, but it also preserves the main additional term, related nontrivially to the original quantum theory. This perturbation-like term is defined by the classical limit of the dispersion characteristics of the quantum theory; it represent the classical dynamics as some effective ‘history of of the world tubes’. The ‘world tubes’ arise in the theory under consideration instead of origination of the world lines in the standard classical mechanics. Our effective ‘fat’ phase trajectories seem really adequate alternative of the ‘thin’ ones if one deals not only with the average quantities, but with their dispersions too.

We have studied in details the symplectic symmetry structure of the generalized classical theory. It is shown, that this structure possesses exactly the matrix-valued General Relativity form. For example, we have found the nonlinear sector of these symmetries and identified it as the matrix-valued Ehlers transformation from the standard Einstein theory. We think, that these established continuous Lee symmetries of the theory can help in construction of the solution of the main dynamical problem, i.e. in integration of the motion equations. It seems possible, that one can obtain a constructive approach to study in details the time irreversible classical dynamics supplemented by the fundamental quantum theory of the type, presented in this work.

1.1 Quantum dynamics and thermodynamics

It seems clear, that any realistic generalization of the quantum theory must start with some complex linear space of state vectors $\Psi_1,\ \Psi_2$, etc., with some well-defined scalar products $\Psi_1^+\Psi_2$, etc., of these vectors. Moreover, one must preserve a probability interpretation of the theory. For example, in the normalizable case one must use the relation

$$P = \frac{\Psi_1^+\Psi_2\ \cdot\ \Psi_2^+\Psi_1}{\Psi_1^+\Psi_1\ \cdot\ \Psi_2^+\Psi_2} \tag{1.1.1}$$

for the probability P to find the system in the condition with the state vector Ψ_1 when this system is specified by the state vector Ψ_2. Then, it is necessary to deal with some set of the observables Q_1, Q_2, etc., which are linear Hermitian operators acting on the space of the state vectors. Without any doubt, the rule for calculation of the average value $\bar{Q}$ for the observable Q, which is related to the state vector Ψ, must save its well-known conventional form:

$$\bar{Q} = \frac{\Psi^+ Q \Psi}{\Psi^+\Psi}. \tag{1.1.2}$$

Also, the average values of the observable quantities must preserve their meaning in relation of the results of the quantum theory to the corresponding observations in the real world.

Then, dynamical aspects of the generalized quantum theory must be described by the help of some evolutionary parameter τ and Hamiltonian operator $\mathcal{H}$. For conservative systems one has $\mathcal{H}_{,\tau} = 0$ and $\Psi = \Psi(\tau)$ in the Schrödinger's picture which we use in this work. The main dynamical equation of the theory also must preserve its conventional Schrödinger's form,

$$i\hbar\Psi_{,\tau} = \mathcal{H}\Psi. \tag{1.1.3}$$

Note, that all elements of the quantum theory listed above are completely standard ones.

Our modification of the quantum theory is based on the use of the complex time parameter $\tau \neq \tau^*$ and non-Hermitian Hamiltonian $\mathcal{H} \neq \mathcal{H}^+$. We consider a holomorphic variant of the theory, with $\Psi_{,\tau^*} = 0$ and $\mathcal{H}_{,\tau^*} = 0$, which leads to the simplest generalization of the standard quantum theory. We restrict our consideration by the theories with $[\mathcal{H},\ \mathcal{H}^+] = 0$; this last relation seems the most natural continuation of the one for the standard theory. Actually, it becomes an identity when $\mathcal{H} = \mathcal{H}^+$. Then, we parameterize the complex evolutionary parameter τ in terms of the real variables t and β in the following way:

$$\tau = t - i\,\frac{\hbar}{2}\,\beta, \tag{1.1.4}$$

whereas the non-Hermitian operator $\mathcal{H}$ will be represented using the Hermitian ones E and Γ as

$$\mathcal{H} = E - i\,\frac{\hbar}{2}\,\Gamma. \tag{1.1.5}$$

Note, that

$$[E,\ \Gamma] = 0, \tag{1.1.6}$$

in accordance to the restriction imposed above. In the next section we will argue that if one relates the quantities t and E to the 'usual' time and energy operator, respectively, then the remaining ones β and Γ mean the inverse temperature $\beta = 1/T$ and the operator of decay parameters of the quantum system. Also, it will be demonstrated that the last operator defines the arrow of time, which seems naturally in view of the transparent irreversibility of the any decay process.

Then, this scheme of generalization of the quantum theory must be completed by introducing of a conception of the thermodynamic regime, which has a form of fixation of the temperature function, i.e. of the relation $\beta = \beta(t)$. For example, one can study evolution of the system in the isothermal case $\beta = \text{const}$; also it is possible to analyze the adiabatic situation with $\bar{E}(t,\beta) = \text{const}$. The most general thermodynamic regime, which is related to some observable quantity Q, can be defined in the following form:

$$f\left(t,\ \beta,\ \bar{Q}(t,\beta)\right) = 0. \tag{1.1.7}$$

Here it is important to stress, that our main goal is related to study of the time arrow effect for class of the quantum theories defined by the relations (1.1.1)–(1.1.5) and the restriction (1.1.7). Below it is shown that this effect actually takes place in the physically well-motivated thermodynamic regimes. Thus, the general theoretical scheme given above provides really natural tool for incorporating of the Second Law of thermodynamics into the pure Hamiltonian quantum theory. Note, that this approach leads not only to the unification of thermodynamics with quantum theory. Also, the unified theory of thermodynamics with classical mechanics (or with classical field theory) can be obtained in the classical limit of the corresponding quantum unification scheme.

1.2 Quantum thermodynamic regimes

So, our main goal is to detect and to study the irreversible aspects of evolution of the generalized quantum system defined in the previous section. To do it, let us express the significant quantities of the theory in terms of the basis of eigenvectors ψ_n of the commuting operators E and Γ. We take it in its orthonormal form, i.e. we mean that the identities $\psi_n^+\psi_k = \delta_{nk}$ take place for all values of the indexes n and k. Here, of course, these indexes must be understood in the appropriate multi-index sense, and all the summations arising below are of the corresponding generalized type.

The eigenvalue problem under consideration reads:

$$E\psi_n = E_n\psi_n, \qquad \Gamma\psi_n = \Gamma_n\psi_n; \tag{1.2.1}$$

it can be reformulated in terms of the non-Hermitian operator $\mathcal{H}$. Actually, it is easy to see, that ψ_n is the eigenvector for this operator, which corresponds to the complex eigenvalue $\mathcal{H}_n = E_n - i\hbar/2\,\Gamma_n$. Then, the state vectors $\Psi_n = e^{-i\mathcal{H}_n\tau/\hbar}\psi_n$ satisfy the Schrödinger's equation (1.1.3), and also form the complete (but τ-dependent) basis. This basis can be used for representation of any solution Ψ of the Schrödinger's equation in the form of linear combination with some set of constant parameters C_n, i.e., as

$$\Psi = \sum_n C_n\Psi_n. \tag{1.2.2}$$

Using this decomposition and the basis properties, let us calculate the probability P_n to find the quantum system in its basic state Ψ_n, when it is described by the state vector Ψ. After the application of Eq. (1.1.1) one obtains, that

$$P_n = \frac{w_n}{Z}, \tag{1.2.3}$$

where $Z = \sum_n w_n$,

$$w_n = \rho_n e^{-(E_n\beta+\Gamma_n t)}, \tag{1.2.4}$$

and $\rho_n = |C_n|^2$. Note, that all the following quantum analysis will be related to the study of dynamics of the probabilities (1.2.3)–(1.2.4) in the different thermodynamical regimes and for the different special realizations of the generalized quantum theory. In this analysis, the formula (1.2.4) provides the base for the both theoretical and experimental study of the our generalization comparing with the standard quantum theory.

First of all, let us consider the evolution of two very special systems which will explain our interpretation of the imaginary part of the complex parameter τ and the anti-Hermitian one of the Hamiltonian $\mathcal{H}$. Namely, the first system has the coinciding eigenvalues Γ_n for the all indexes n. It is easy to see, that $w_n = \rho_n e^{-E_n\beta}$ in this case. This means, that the quantity β is actually the inverse absolute temperature if E_n had been identified as the n-th energy level of the system. The second special system will be specified by the coinciding eigenvalues E_n. Then $w_n = \rho_n e^{-\Gamma_n t}$, so the quantities Γ_n have the sense of the decay parameters, if t means the conventional ('usual') time. Actually, let us consider, for example, the system with $\Gamma_{n_\star} = \min_n\{\Gamma_n\} = 0$ in the situation where the single level n is weakly excited under the level $n_\star$. In this special case $\rho_m = \rho_m \delta_{mn}$, where $n, m \neq n_\star$, and also $\rho_{n_\star} \approx 1$ whereas $\rho_m \approx 0$. It is easy to see, that for this quantum state $P_m \approx \rho_m e^{-\Gamma_m t}$, so $t_m = 1/\Gamma_m$ is a conventional 'time of life' of the basis state which has the energy E_m.

Note, that in the same situation with $\Gamma_{n_\star} = \max_n\{\Gamma_n\}$, one deals with the exponentially increasing probability $P_m(t)$. However, we use the term 'parameters of decay' for the quantities Γ_n for all regimes of the evolution. Then, it is clear that the solution space of the theory of the discussing type can be decomposed into the direct sum of the solution subspaces which have a given value of the energy or of the parameter of decay. For all these subspaces the interpretation of β and Γ_n is the same as for the special systems of the first and of the second types discussed above, respectively. In fact, we extend the interpretation of these physical quantities (as well as the interpretation for the quantities E and t) to the total solution space of the generalized quantum theory, making a simple and really natural hypothesis.

Now let us consider the non-specified quantum theory of the form, presented in the previous section, and fix the isothermal regime of its thermodynamical evolution. It is easy to prove, that in the case of $\beta = \text{const}$, the dynamical equations for the basic probabilities read:

$$\frac{dP_n}{dt} = -\left(\Gamma_n - \bar{\Gamma}\right) P_n. \tag{1.2.5}$$

To perform the analysis, let us study a behavior of the quantity $\bar{\Gamma}$. After some calculations one obtains, that

$$\frac{d\bar{\Gamma}}{dt} = -D^2_\Gamma, \tag{1.2.6}$$

where $D^2_\Gamma = \overline{\left(\Gamma - \bar{\Gamma}\right)^2}$ is the squared dispersion of the quantum observable Γ. From Eq. (1.2.6) it follows, that the function $\bar{\Gamma}(t)$ is not increasing, so the isothermal regime actually allows the time arrow which can be naturally related to the average value of the decay operator Γ.

Then, the probability P_n rises when $\bar{\Gamma} > \Gamma_n$ and degenerates if $\bar{\Gamma} < \Gamma_n$. Thus, in the isothermal regime the quantity $|\Gamma_n - \bar{\Gamma}|$ has a sense of logarithmic decrement of the growth or degeneration of the probability to find the quantum system on the energy level

E_n. Also, in this regime one obtains the following picture for asymptotics of the probabilities at $t \to +\infty$: all the 'activated' (i.e., with $\rho_n \neq 0$) probabilities with the maximal value of the decay parameters rise droningly, whereas all the remaining probabilities fall to the zero ones. Namely, let us define the multi-index $n_\star$ by the relation $\Gamma_{n_\star} = \min_n\{\Gamma_n\}$, again.

Then, for the only non-degenerating probabilities $P_{n_\star}(t)$ one obtains the following asymptotical result:

$$P_{n_\star}(+\infty) = \Pi P_{n_\star}(0), \tag{1.2.7}$$

where the scale parameter $\Pi > 1$ reads:

$$\Pi = 1 + \frac{\sum_{n \neq n_\star} \rho_n e^{-E_n \beta}}{\sum_{n_\star} \rho_{n_\star} e^{-E_{n_\star} \beta}}. \tag{1.2.8}$$

Note, that the relations (1.2.7)–(1.2.8) have the form of a 'dressing procedure' in the standard quantum field theory. This circumstance seems really hopeful for renormalization of the quantum field theories with non-Hermitian Hamiltonian and complex time parameter. Actually, in the standard quantum field theory one deals with the oscillating harmonics (in the corresponding representation on shell), which cannot be eliminated without fail of the standard mathematical logic. However, one needs in cut of the increasing modes to reach the theoretical scheme with finite calculations. It is a well known fact, that all known cut procedures are in contradiction with all 'normal neglecting principles'. The generalized quantum theory under consideration allows one to speak about the modes which degenerate dynamically, and also about the modes which remain 'alive' at the 'big times'. Moreover, these last modes of the exact quantum theory solutions become renormalized, if one compares their initial and final probabilities, see Eqs. (1.2.7)–(1.2.8).

Then, it is easy to prove, that in the general case of $\beta = \beta(t)$, the dynamical equations for the basic probabilities have the following form:

$$\frac{dP_n}{dt} = -\left[\Gamma_n - \bar{\Gamma} + \left(E_n - \bar{E}\right)\frac{d\beta}{dt}\right] P_n. \tag{1.2.9}$$

Here the specific function $d\beta/dt$ must be extracted from the corresponding thermodynamical regime (1.1.7). In the adiabatic case, when $\bar{E} = \sum_n E_n P_n = \text{const}$, one obtains immediately that

$$\frac{d\beta}{dt} = -\frac{\overline{E\Gamma} - \bar{E}\bar{\Gamma}}{D_E^2}, \tag{1.2.10}$$

where D_E denotes the dispersion of the energy operator E. Using Eqs. (1.2.9)–(1.2.10), for the dynamics of the quantity $\bar{\Gamma}$ in the adiabatic regime one obtains:

$$\frac{d\bar{\Gamma}}{dt} = -D_\Gamma^2 \left[1 - \frac{\left(\overline{E\Gamma} - \bar{E}\bar{\Gamma}\right)^2}{D_E^2 D_\Gamma^2}\right]. \tag{1.2.11}$$

Our goal is to show that the function $\bar{\Gamma}(t)$ is non-increasing again, so the generalized quantum system possesses the well-defined arrow of time in its adiabatic regime of evolution as well as in the isothermal case. To do it, let us prove that the expression [...] in Eq. (1.2.11) is not negative. Let us introduce the formal vector quantities X and Y with the components $X_n = E_n - \bar{E}$ and $Y_n = \Gamma_n - \bar{\Gamma}$, respectively, and with scalar product $(XY) = \sum_n P_n X_n Y_n$. It is easy to see, that in terms of these quantities [...] $= 1 - (XY)^2/[(XX)(YY)]$, so this expression is really non-negative in view of the general Cauchy-Buniakowski inequality.

At the end of this section we would like to note, that the function $\bar{\Gamma}(t)$ can not provide the arrow of time for the arbitrary regime of thermodynamic evolution of the generalized quantum system. Actually, this statement becomes transparent, if one considers the specific regime with $\bar{\Gamma} = \text{const}$. It is clear, that the study of the corresponding dynamics must be performed in some completely different terms. In fact, we prove its irreversible character using the entropy function for one special class of the generalized quantum systems in the next section.

1.3 Example I: Thermodynamics of quantum subsystem

Let us consider the theory of the discussing type with $n = (0, \nu)$, $\Gamma_0 = \gamma \neq 0$, and $\Gamma_\nu = 0$ for all values of the collective index ν. Our nearest goal is to study the nontrivial thermodynamical regime $\bar{\Gamma} = \text{const}$ of the evolution of this system in the time arrow framework.

First of all, we rewrite the basic probabilities of the theory, i.e., the quantities

$$\begin{aligned} P_0 &= \frac{\rho_0}{\rho_0 + e^{\gamma t}\tilde{Z}}, \\ P_\nu &= \frac{e^{\gamma t}\tilde{w}_\nu}{\rho_0 + e^{\gamma t}\tilde{Z}}, \end{aligned} \tag{1.3.1}$$

where $\tilde{Z} = \sum_\nu \tilde{w}_\nu$, $\tilde{w}_\nu = \rho_\nu e^{-\tilde{E}_\nu \beta}$ and $\tilde{E}_\nu = E_\nu - E_0$, in terms of the average magnitude $\bar{\Gamma}$. The result reads:

$$P_0 = \frac{\bar{\Gamma}}{\gamma}, \qquad P_\nu = \left(1 - \frac{\bar{\Gamma}}{\gamma}\right)\tilde{P}_\nu, \tag{1.3.2}$$

where $\tilde{P}_\nu = \tilde{w}_\nu/\tilde{Z}$. Thus, in the case of $\bar{\Gamma} = \mathrm{const}$, one has $P_0 = \mathrm{const}$ and $P_\nu \sim \tilde{P}_\nu$, so the dynamics of the system under consideration is defined by the dynamics of its effective subsystem with the basic probabilities $\tilde{P}_\nu$ completely.

Note, that this subsystem, which is characterized by the energies $\tilde{E}_\nu$, their average value $\bar{\tilde{E}} = \sum_\nu \tilde{P}_\nu \tilde{E}_\nu$, and the squared dispersion $D^2_{\tilde{E}} = \overline{\left(\tilde{E} - \bar{\tilde{E}}\right)^2}$, is the conventional thermodynamic system. Its evolution is given by the temperature regime $\beta = \beta(t)$, which can be extracted from the corresponding macroscopic condition ($\bar{\Gamma} = \mathrm{const}$ in the situation under consideration). For such systems the dynamical equations read:

$$\begin{aligned} \tilde{P}_{\nu,t} &= \beta_{,t}\left(\bar{\tilde{E}} - \tilde{E}_\nu\right)\tilde{P}_\nu, \\ \bar{\tilde{E}}_{,t} &= -\beta_{,t} D^2_{\tilde{E}}, \end{aligned} \tag{1.3.3}$$

and the arrow of time can be related to the both functions $\beta(t)$ and $\bar{\tilde{E}}(t)$. Moreover, one can introduce the entropy $\tilde{S} = -\ln \tilde{\mathrm{Z}} - \beta\bar{\tilde{\mathrm{E}}}$, which growth is proportional to the one for the average energy:

$$\tilde{S}_{,t} = -\beta \bar{\tilde{E}}_{,t}. \tag{1.3.4}$$

However, in this chapter we use the quantity $\bar{\tilde{E}}(t)$ as the Lyapunov function as the most natural tool in the analysis of the dynamics of the system under consideration.

It is easy to prove, that in the case of $\bar{\Gamma} = \mathrm{const}$, the temperature regime is defined by the relation

$$t = \frac{1}{\gamma}\left\{\ln\left[\rho_0\left(\frac{\gamma}{\bar{\Gamma}} - 1\right)\right] - \ln\tilde{\mathrm{Z}}\right\} \tag{1.3.5}$$

(note, that $\tilde{Z} = \tilde{Z}(\beta)$ here). From this relation and Eq. (1.3.3) it follows, that

$$\bar{\tilde{E}}_{,t} = -\frac{\gamma}{\bar{\tilde{E}}} D^2_{\tilde{E}}, \tag{1.3.6}$$

so the function $\bar{\tilde{E}}(t)$ defines the arrow of time in the all regions of conservation of its sign.

In view of this fact, it is natural to restrict our consideration by the systems which possess minimal or maximal value of their energy spectrum. Namely, it is clear, that $\bar{\tilde{E}} > 0$ if $E_0 = \mathrm{min_n}\left\{\mathrm{E_n}\right\}$ (the situation A), and $\bar{\tilde{E}} < 0$ if $E_0 = \mathrm{max_n}\left\{\mathrm{E_n}\right\}$ (the situation B). Also, let us call the theories with $\gamma < 0$ and $\gamma > 0$ as the cases I and II, respectively. Then, the class of theories under consideration splits into the four subclasses IA, IB, IIA and IIB –

in accordance with the sign of their decay parameter γ, and with the 'min/max' character of the energy level E_0. Finally, it is seen that the theories IA and IIA are characterized by increasing of their average energy $\bar{\tilde{E}}$, whereas the theories IB and IIB correspond to the decreasing behavior of this quantity. It is useful to note, that $|\bar{\tilde{E}}|$ is the Lyapunov function for the theory under consideration for the all types of the spectrum $\{E_n\}$ in the thermodynamical regime $\bar{\Gamma} = \mathrm{const}$, as it follows immediately from Eq. (1.3.6). However, the quantity $\bar{\tilde{E}}$ seems more physically motivated function in view of its close relation to the entropy, see Eq. (1.3.4).

Now let us study the adiabatic regime of evolution of the quantum system under consideration. First of all, we rewrite the basic probabilities (1.3.1) in terms of the quantity $\bar{E}$, i.e. using the average energy of the system. The result reads:

$$P_0 = 1 - \frac{\bar{\mathcal{E}}}{\bar{\tilde{E}}}, \qquad P_\nu = \frac{\bar{\mathcal{E}}}{\bar{\tilde{E}}}\tilde{P}_\nu, \tag{1.3.7}$$

where $\bar{\mathcal{E}} = \bar{E} - E_0$. Thus, the dynamics of the total system is defined by the dynamics of its effective subsystem (with the probabilities $\tilde{P}_\nu$) again. From Eq. (1.3.7) it follows, that

$$0 \le \frac{\bar{\mathcal{E}}}{\bar{\tilde{E}}} \le 1; \tag{1.3.8}$$

this double inequality defines the region of possible values of the temperature parameter β after imposing of the condition $\bar{E} = \mathrm{const}$. The temperature regime in this case is given by the relation

$$t = \frac{1}{\gamma}\ln\left[\frac{\rho_0\bar{\mathcal{E}}}{\tilde{\mathrm{Z}}\left(\bar{\tilde{\mathrm{E}}} - \bar{\mathcal{E}}\right)}\right]. \tag{1.3.9}$$

It defines the inverse temperature evolution from the initial value $\beta_0 = \beta(0)$, where

$$\bar{\tilde{E}}(\beta_0) = \bar{\mathcal{E}}\left(1 + \frac{\rho_0}{\tilde{Z}(\beta_0)}\right), \tag{1.3.10}$$

to the final one β_{as}, where

$$\bar{\tilde{E}}(\beta_{\mathrm{as}}) = \bar{\mathcal{E}}. \tag{1.3.11}$$

From Eq. (1.3.9) it follows that the system actually achieves the final value of its temperature at the time asymptotics $t \to +\infty$; this opportunity explains the notations used.

Then, taking into account Eqs. (1.3.6) and (1.3.9), it is easy to prove that

$$\bar{\tilde{E}}_{,t} = -\gamma\left(\bar{\tilde{E}} - \bar{\mathcal{E}}\right)\frac{D^2_{\tilde{E}}}{\overline{\tilde{E}^2} - \bar{\tilde{E}}\bar{\mathcal{E}}}. \tag{1.3.12}$$

From the inequality (1.3.8) it follows, that $\overline{\tilde{E}^2} - \bar{\tilde{E}}\bar{\mathcal{E}} \geq D^2_{\tilde{E}}$, so the sign of $\bar{\tilde{E}}_{,t}$ is opposite to the sign of the quantity $\gamma\left(\bar{\tilde{E}} - \bar{\mathcal{E}}\right)$. Thus, it is natural to consider two regions $\bar{\tilde{E}} > \bar{\mathcal{E}} > 0$ and $\bar{\tilde{E}} < \bar{\mathcal{E}} < 0$ of the possible temperature evolution, which are two branches of the region defined by inequality (1.3.8). Then, these two regions correspond exactly to the situations A and B for the thermodynamical regime $\bar{\Gamma} = \mathrm{const}$ studied above. Using this opportunity, we can realize these two situations by consideration of the theories with $E_0 = \min\{\mathrm{E_n}\}$ and $E_0 = \max\{\mathrm{E_n}\}$, respectively, again. Finally, it is easy to see, that behavior of the Lyapunov function $\bar{\tilde{E}}(t)$ in the both thermodynamical regimes ($\bar{\Gamma} = \mathrm{const}$ and $\bar{E} = \mathrm{const}$) is the same for the all four variants of the theory (IA, IB, IIA and IIB).

At the end of the analysis of the adiabatic regime we would like to stress, that the quantity $\bar{\Gamma}$ is the decreasing Lyapunov function for all signs of the decay parameter γ and for all types of the energy spectrum E_n. Namely, one can check, that

$$\bar{\Gamma}_{,t} = -\gamma^2\frac{\bar{\mathcal{E}}}{\bar{\tilde{E}}}\left(1 - \frac{\bar{\mathcal{E}}}{\bar{\tilde{E}}}\right)\frac{D^2_{\tilde{E}}}{\overline{\tilde{E}^2} - \bar{\tilde{E}}\bar{\mathcal{E}}}; \tag{1.3.13}$$

so $\bar{\Gamma}_{,t} \leq -\gamma^2\frac{\bar{\mathcal{E}}}{\bar{\tilde{E}}}\left(1 - \frac{\bar{\mathcal{E}}}{\bar{\tilde{E}}}\right) \leq 0$ in the complete accordance to the general results obtained for the adiabatic case in the previous section. It is clear, that both total quantum system and its quantum subsystem considered above have the real physical sense and are really interesting for all possible applications. Here we would like to note, that one can generalize the results of this section to the case of set of the quantum subsystems which form together the closed total quantum system of the discussing type.

1.4 Example II: Left-right asymmetry and time arrow

In this section we consider a system, which consists of two parts again (its 'left' and 'right' constituents) – in accordance to specification of the symmetry operator Γ taken in this case. Namely, we define it as proportional to some parity operator; this leads to the double realization $\psi_{\pm n}$ for the all energy levels E_n . The corresponding eigenvalue problem reads:

$$E\,\psi_\pm = E_n\,\psi_\pm, \qquad \Gamma\,\psi_\pm = \mp\gamma\,\psi_\pm; \tag{1.4.1}$$

we take the common basis $\{\psi_\pm\}$ of the energy and decay operators in the orthonormal form (i.e. below we mean that $\psi^+_{\pm\, n_1}\psi_{\pm\, n_2} = \delta_{n_1 n_2}$ and $\psi^+_{+\, n_1}\psi_{-\, n_2} = 0$ for the all indexes n_1, n_2). For definiteness, we name the eigenvectors $\psi_{-\,n}$ and $\psi_{+\,n}$ as the 'right' and 'left' ones, respectively, and take $\gamma > 0$.

Then, it is clear that the τ-dependent basis vectors $\Psi_\pm = e^{-i\mathcal{H}_{\pm n}\tau/\hbar}\psi_\pm$, where $\mathcal{H}_{\pm n} = E_n \pm i\frac{\hbar}{2}\gamma$, satisfy the Schrödinger's equation identically. Let us consider an arbitrary solution Ψ of this equation; it can be represented as some linear combination $\Psi = \sum_{\pm\, n} C_{\pm n}\Psi_{\pm n}$ with the constant coefficients $C_{\pm n}$ in respect to this basis. Using the projection relation given above, one can calculate the probabilities $P_{\pm n}$ to find the system which is situated in the state Ψ in the energy eigenstates $\Psi_{n\,\pm}$. The result reads:

$$P_{\pm n} = \frac{w_{\pm n}}{Z}, \tag{1.4.2}$$

where $w_{\pm n} = e^{\pm\gamma t}\,\tilde{w}_{\pm n}$, $Z = e^{\gamma t}\tilde{Z}_+ + e^{-\gamma t}\tilde{Z}_-$; and also

$$\tilde{w}_{\pm n} = \rho_{\pm n} e^{-E_n\beta}, \qquad \tilde{Z}_\pm = \sum_n \tilde{w}_{\pm n}, \tag{1.4.3}$$

where $\rho_{\pm n} = |C_{\pm n}|^2$.

Our goal is to study evolution of the 'left' and 'right' parts of the system, which we relate to subsets of the probabilities $P_{-\,n}$ and $P_{+\,n}$, respectively. It is easy to check, that this evolution can be expressed in terms of the dynamics of effective subsystems with the probabilities

$$\tilde{p}_{\pm\, n} = \frac{\tilde{w}_{\pm\, n}}{\tilde{Z}}, \tag{1.4.4}$$

which have the standard β-dependent statistical form in view of Eq. (1.4.3). Note, that in the isothermal case (with $\beta = \beta(t) = \text{const}$), one obtains immediately, that

$$\lim_{t\to\pm\infty} P_{\pm\, n}(t) = \tilde{p}_{\pm\, n}, \qquad \lim_{t\to\pm\infty} P_{\mp\, n}(t) = 0. \tag{1.4.5}$$

Thus, in the isothermal evolution this quantum system transforms from its 'left' realization to the 'right' one. As the result,

$$\lim_{t\to\pm\infty}\bar{\Gamma}(t) = \pm\gamma, \tag{1.4.6}$$

i.e. total shift of the average value of the decay operator Γ is equal to 2γ. It is clear, that the same situation takes place for the arbitrary thermodynamic regime, which admits the asymptotic inverse temperatures $\beta_\pm = \lim_{t\to\pm\infty}\beta(t)$. In this case one must replace $\tilde{p}_{\pm\,n}$ by $\tilde{p}_{\pm\,n}(\beta_\pm)$ in Eq. (1.4.5).

Now let us study in details a special case where the quantum system reaches a total symmetry between its left and right constituents at some finite time $t_\star$. Namely, we are interesting in the dynamics with $P_{+\,n}(t_\star) = P_{-\,n}(t_\star)$ for all values of the collective index n. Taking $t_\star = 0$, one obtains the system with $\rho_{+\,n} = \rho_{-\,n} \equiv \rho_n$, so

$$P_{\pm\,n} = \frac{\tilde{p}_n}{1+e^{\mp 2\gamma t}}, \tag{1.4.7}$$

where $\tilde{p}_n = \tilde{w}_n/\tilde{Z}$, $\tilde{Z} = \sum_n \tilde{w}_n$, and $\tilde{w}_n = \rho_n e^{-E_n\beta}$. Thus, in this special case one deals with the single effective system with set of the β-dependent probabilities $\tilde{p}_n$. Then, for the total system under consideration the average value of the decay operator is

$$\bar{\Gamma} = \gamma\ \tanh\ \gamma t, \tag{1.4.8}$$

so the quantity $\bar{\Gamma}(t)$ plays a role of the monotonously increasing Lyapunov function for the attractor 'right' state, and detects the dynamical parity breaking in the system. This function defines the universal arrow of time, it demonstrates the irreversible character of the system dynamics in the really transparent form.

Note, that form of this function does not depend on the concrete temperature regime $\beta = \beta(t)$. Also, it is interesting to stress, that the average energy $\bar{E}$ of the total system coincides with the same quantity for the effective subsystem $\bar{\tilde{E}} = \sum_n \tilde{p}_n\, E_n$. Thus, in this special case the isothermal regime $\beta = \mathrm{const}$ and the adiabatic regime $\bar{E} = \mathrm{const}$ become identical. To consider another regime of the evolution, one can take, for example, the symmetry operator Q with 'left' and 'right' eigenvalues Q_{-n} and Q_{+n} which does not coincide identically. Then, the thermodynamic regime with $\bar{Q} = \mathrm{const}$ defines the 'temperature curve' with

$$t = \frac{1}{2\gamma}\ln\left(\frac{\bar{Q}\ -\bar{\tilde{Q}}_-}{\bar{\tilde{Q}}_+ - \bar{Q}}\right), \tag{1.4.9}$$

where $\bar{\tilde{Q}}_\pm = \bar{\tilde{Q}}_\pm(\beta) = \sum_n Q_{\pm\,n} P_{\pm\,n}(\beta)$. For this regime $\bar{\tilde{E}}_{,t} = -\beta_{,t} D^2_{\tilde{E}} \neq 0$, where $D_{\tilde{E}}$ is a dispersion of the energy for the effective subsystem introduced above. It is clear, that one can use not only the symmetry operators for fixing and study of non-isothermal thermodynamic regimes. Actually, it is possible to relate such regimes to average values of the observables which does not commute with the Hamiltonian of the system under consideration.

One important realization of the generalized quantum theory is related to theory of the massless Dirac field. Namely, it is easy to see, that the modified Dirac equation

$$i\hbar\,\Psi_{,\tau} = \left(\vec{\alpha}\,\vec{p} - i\,\frac{\hbar}{2}\,\gamma\,\gamma_5\right)\Psi \tag{1.4.10}$$

has the standard form with the Dirac energy operator $E = \vec{\alpha}\,\vec{p}$ (where $\vec{p}$ is the momentum operator), and the parity decay operator $\Gamma = \gamma\,\gamma_5$. Here $\alpha_k = \gamma_0\gamma_k \quad (k = 1, 2, 3)$, and the Hamiltonian commutation relation is satisfied obviously. Thus, in complete accordance with the general results presented above, the originally mixed left-right massless Dirac quantum system transforms to its strictly polarized asymptotic state in the case of $\gamma \neq 0$. One can choose the sign of this free constant parameter to obtain appropriate helicity of the final state to identify it with the real neutrino system.

Actually, it is a well known fact, that the real neutrinos are 'left' (whereas antineutrinos are 'right'), and this circumstance seems intriguing in view of its unclear and, moreover, 'pure random' status in the standard particle physics. We think, that this our new approach, which provides a dynamical solution for the helicity asymmetry problem, is more natural than fundamentally asymmetric standard scheme. Also, the new approach relates the parity violation in the real neutrino system with the arrow of time in its evolution (and with the second law of thermodynamics after all). Thus, in framework of the generalized quantum theory the 'random' feature of the real quantum physical system becomes a consequence of its irreversible history from the universal thermodynamic point of view.

1.5 Discussion

In this part we have developed new thermodynamic generalization of the quantum theory which possesses a well-defined arrow of time in the most important regimes of its thermodynamic evolution. This generalization is based on the use of non-Hermitian Hamiltonian, which commuting Hermitian and anti-Hermitian parts define the energy operator and minus half of the operator of decay parameters of the energy ejgenstates, respectively. Also, this modified quantum theory deals with the complex time parameter, which real part coincides with the 'usual' time, whereas the imaginary one is identified as minus half of the inverse absolute temperature. We postulate strictly analytic dependence of state vectors on the complex time (in the Schrodinger's picture), and consider the generalized conservative systems with time- and temperature-independent Hamiltonian.

This pure Hamiltonian scheme is completed by introducing of the thermodynamic regime of evolution, which must be defined in the terms of average value of the corresponding observable quantity. In fact, this fixation of the thermodynamic regime means 'incorporating' of the macroscopic observer into the originally microscopic framework of the modified quantum theory. Constructively, every consistent thermodynamic regime determines some evolution of temperature of the system, i.e., it defines some temperature curve on the complex plane of the parameter of evolution. Our main goal is related to search and study of the irreversibility effects of the corresponding quantum dynamics. In this paper we have studied in details the isothermal and adiabatic regimes of evolution; we have shown that they actually possess the well-defined arrow of time.

Also, we have studied the problem of thermodynamic evolution of the quantum subsystem. Namely, we have decomposed the modified quantum system into two parts - the subspace with extreme value of energy (i.e., vacuum for the case of minimal energy), and the subsystem (the quantum subsystem mentioned above) which is constructed from the all remaining energy eigenstates. We have analyzed the situation when this subsystem (as a single whole system) and the state with extreme energy are characterized by different times of the decay. We have established, that the average energy of the subsystem, as well as its conventional entropy, are the monotonous functions; they actually define the arrow of time in the, for example, adiabatic regime of evolution of the original quantum system.

Also, we have formulated new and concrete mechanism of 'insertion' of the second law of thermodynamics into the standard quantum theory. Namely, it is shown, that to guarantee the functioning of this law, one needs only in the adding of the positive parameter of the decay to the vacuum state of the theory. This simple mechanism gives actually promising general scheme of modernization of the all quantum physics.

2 Generalized classical theory

The main goal of this part is to develop a consistent classical limit of the quantum theory presented and studied above. This new classical theory must possess the main property of the original quantum theory – the essential irreversibility of its dynamics. Thus, it must be a classical mechanics (or/and a classical wave theory) unified with a classical thermodynamics. We derive it, using asymptotical approach of the Laplace type to calculation of the classical limit of the average values of the quantum variables and all possible correlations of them.

At the first time, we establish the consistent classical dynamics of quantum numbers (like the quantum numbers n, l and m in the quantum solution of the Kepler problem), which possesses all the fundamental properties of the theory in its general quantum regime of evolution. In terms of these numbers, as it is shown below, the Hamiltonian structure of the theory reaches the simplest form. Namely, it is easy to understood, that if one considers the quantum numbers as a possible set of the canonical moments, one deals with the representation which Hamiltonian is coordinate-independent. In fact, this means study of the problem using some modification of the Hamilton-Jacoby approach: this representation is based on the use of the so called 'action-angle variables', which form the most convenient set of the canonical coordinates and moments.

In our construction of the classical limit of the quantum theory we start with the extremal (variational) principle. We define the classical dynamics as the dynamics of the resonance structures in the probability density distributions. Namely, we consider the evolution of well-defined maximums ('resonances') of the probability density, which were formed at the initial time and cut terms depending on the Planck constant $\hbar$ (in 'hard' classical regime). We would like to stress, that our classical limit of the modified quantum theory considered in the first part has two independent infinitesimal parameters: the resonance width ε and the Planck constant $\hbar$. In fact, the first parameter defines the classical kinematical objects (these 'resonances' themselves) in the general quantum framework, whereas vanishing of the second one maps the general quantum dynamics into its special classical regime. Note, that one of our goals is to save a control under the dissipation/concentration of the probability density resonances in the classical theory framework. We are interesting in building of classical dynamics which allows one to deal with some remnants of the quantum uncertainty principle and its various consequences.

From the pure kinematical point of view, we have established, that the convenient uncertainty relations can be incorporated correctly into the general structure of the classical theory. These fundamental relations are expressed in terms of the classical limit of the correlations between the canonical coordinates and moments. Also, we have pointed out, that the correlations under discussion are the analogies of the metric coefficients from the General Relativity. They describe the curvature effects in dynamics of the classical objects ('resonances') in the curved phase space. Note, that it arises a possibility to identify the classical gravitation as the quantum effect of the dispersion type.

From the pure dynamical point of view, it is shown, that the classical evolution of the system is really time-irreversible. Namely, we have proved the presence of the arrow of time in the same thermodynamical regimes as it had been done in the quantum theory case. Firstly, we have established this irreversibility in terms of the action-angle variables, and, after that, we have expanded the result to the arbitrary canonical variables case. We have detected the arrow of time not only in the 'hard' classical limit of the original quantum theory defined above (with $\hbar = 0$). Moreover, we have proved irreversibility of the theory dynamics in the case of $\varepsilon \to 0$ and free constant parameter $\hbar$ (the 'soft' classical theory). This 'soft' regime seems really promising in context of the quasi-classical gravity incorporation into the theory under consideration.

For the 'soft' variant of the classical theory, it is established, that the correlations can be unified into the single null-curvature matrix field of the symplectic type. The corresponding symplectic structure of the theory remains conserved under the action of the symplectic group of the continuous canonical transformations. This group coincides with matrix-valued group of the hidden symmetries of the standard General Relativity. We have estableshed, that the generalized Hamilton equations describing our theory in its 'soft' regime also possess the hidden symplectic symmetry.

Then, we have studied and classified these canonical symplectic maps, and extract a non-linear sector of transformations of the Ehlers type from them. Using one special established discrete map, we have detected a 'duality' between the corpuscular and wave classical sectors in the general 'soft' classical theory framework under consideration. Finally, we have proposed, how one can simplify the constructing and the following study of dynamics of the resulting classical theory by the help of canonical transformations found.

2.1 Modified classical dynamics in action-angle variables

The main goal of this section is to show, that in the best physically motivated quantum theory with $[\mathcal{H}_1,\ \mathcal{H}_2] = 0$ one has not any problem with derivation of the motion equations, which describe its classical regime in the consistent form. In the other words, this means a possibility of construction of the (modified) classical dynamics, which corresponds the (generalized) quantum theory studied above. This dynamics must possess an arrow of time and will have not any artificial restrictions on the Hamiltonians of the system (like quadratic structure in respect to the canonical variables, etc.).

For our goals, it is convenient to use the notation $w(t,n)$, $p(t,n)$, etc., for the all t_α- and n_k-dependent dynamical variables. For example, the energy 'levels' and the inverse times of decay (the logarithmic decrements of the states) will be written now as $E(n)$ and $\gamma(n)$ (in view of their time-independent nature, see the part 1). Thus, in the general quantum regime of the theory under consideration, one deals with the probabilities

$$p(t,n) = \frac{w(t,n)}{\mathcal{Z}}. \tag{2.1.1}$$

Here the scale factor is defined as $\mathcal{Z}(t) = \sum_n w(t,\ n)$, whereas

$$w(t,\ n) = \rho(n) \exp\left[-\left(E(n)\beta + \Gamma(n)t\right)\right], \tag{2.1.2}$$

and the functions $\rho(n)$ are the corresponding weight parameters. Our main idea is to use the set of quantum numbers n as the new and the most natural set of dynamical variables (the canonical moments) for this system to detect the classical regime of its evolution in the most simple and explicitly consistent form.

At the first time, let us stress again, that the collective parameter n is the N-dimensional set of the quantum numbers for the theory with N degrees of freedom. These quantum numbers are the integer-valued quantities defined by the corresponding stationary Schrodinger problem. Of course, we mean some choice of the complete set of the commuting quantum variables taken (which include both the Hamiltonian operators), for the correct fixing of the all indexes n_k, $k = 1, ..., N$. Our approach to constructing of the classical limit of the quantum theory under consideration coincides with conventional understanding of the quantum numbers in the classical limit of the standard quantum theory. Namely, we define the classical region of the quantum theory as the one with $n >> 1$, where the variable n can be understood as the continuous quantity with a really high approximation. In particular, we will differentiate in respect to the constituents of this collective variable, i.e. in respect to the set of the new canonical variables n_k. In doing so, we are taking into account some well-defined infinitesimal physical parameters of the theory, which enter into the corresponding relations, being multiplied to the variations of the originally discrete variables n_k.

Of course, our definition of the classical regime of the quantum theory is not based on the continualization of the parameter n only. Also, we suppose the presence of the well-defined resonance structure for the probability density in the representation taken (a single 'probability resonance', for the definiteness). This resonance means, for example, the localization of the object in the physical space (if one uses the spatial coordinate representation), and the well-defined wave pocket (for the case of the momentum representation).

Thus, we are interesting in the study of the classical dynamics of quantum states with the resonance structure of the probability density in the space of the collective parameter n. Note, that it is really important to save the corresponding dispersion structure of the theory: in the rough limit without any dispersions in respect to the collective parameter n, one deals with 'pure energetic solutions' of the Schrodinger's equation. It is easy to see, that they lead to the probabilities $P(n)$ independent of the evolutionary parameters t and β. Now let us remember, that these parameters have the sense of the 'usual' physical time and the inverse absolute temperature, respectively, as it was shown in the part 1. However, such situation is not of any interest in context of the study of arrow of time problematic. Below we show, that taking into account of these dispersion parameters in the classical regime of the quantum theory allows one to save all the irreversible quantum effects in the framework of new consistent scheme for the classical dynamics.

Now let us rewrite the weight coefficient $\rho(n)$ in the exponential form,

$$\rho(n) = \exp\left(-\sigma\left(n\right)\right). \tag{2.1.3}$$

Then, for the probability functions one obtains the following representation: $p(t, n) =$

$w(t,n)/\mathcal{Z}$, where $\mathcal{Z} = \int dn \; w(t,n)$, and

$$w(t,n) = \exp\left(-\mathcal{S}_2\right) \tag{2.1.4}$$

with

$$\mathcal{S}_2 = \sigma + \sum_\alpha \tilde{E}_\alpha \tilde{t}_\alpha. \tag{2.1.5}$$

Here we have put $\tilde{t}_1 = t$, $\tilde{t}_2 = \beta$, and $\tilde{E}_1 = \Gamma$, $\tilde{E}_2 = E$ to stress a possibility of generalization of the corresponding dynamics to the case of the arbitrary time dimension. Actually, it will be seen, that all the relations in this dynamics hold their form for any value of the 'time dimension' A (where $\alpha = 1, ...A$). Of course, the hidden complex nature of the real physical time takes place in the original case of $A = 2$ only.

Then, let us denote a 'maximum point' of the probability density for the solution under consideration as n_c, i.e.,

$$P\left(t,\; n_c\right) = P_{\max} = \max_n P\left(t,\; n\right). \tag{2.1.6}$$

Here it is important to stress, that below we study only solutions which possess well-defined 'maximum point' for the any time moment t_α. It is clear, that this 'point' will be some function of this collective time parameter t_α, i.e., one deals with $n_c = n_c(\tilde{t}_\alpha)$. We relate naturally a 'position' of the classical object with this extremal value of the parameter n. Thus, we identify the classical theory with the theory which describes the dynamics of resonance structures of the probability density in the original quantum theory. In the 'most hard' classical limit they are described by the delta-functional distributions.

It is clear, that the highly-resonance classical character of the probability density must be guaranteed by corresponding choice of the weight parameters $\sigma(n)$. Actually, these parameters are some initial data of the original quantum state, so they can be chosen in the appropriate way at the beginning of the evolution of the system. We plan to discuss this choice later. Here the most important goal is to write down the defining relations for the 'classical position' n_c of the object under consideration in the constructive form. Thus, we must derive the modified Hamiltonian equations in the quantum theory motivated and consistent form. The last demanding is not trivial: the equations waited will be partial differential equations of the first order (we have A independent dynamical variables in the theory). This means, that the rough cut of the starting quantum problem can destroy consistency of the dynamical system under consideration.

We perform the derivation of the modified Hamiltonian equations in the n-variables by the help of differentiation of the conventional extremum relation in respect to the total set of times $\tilde{t}_\alpha$:

$$\frac{d}{d\tilde{t}_\alpha}\left(\frac{\partial \mathcal{S}_2\left(\tilde{t}_\alpha,\, q\right)}{\partial q_m}\right)_c = 0, \tag{2.1.7}$$

where the index c means the substitution $q = q_c$ into the differentiation result. Note, that the $\tilde{t}_\alpha$-differentiation is understood here in respect to the total $\tilde{t}_\alpha$-dependence of the corresponding quantity (i.e. with taking into account the hidden dependence $n_c = n_c(\tilde{t}_\alpha)$ in this equation). It fact, Eq. (2.1.7) means a conservation of the extremal character of the classical trajectory of the system during its possible physical evolution. Actually, the quantity in the brackets in (2.1.7) is equal to zero for the 'maximum point' of the probability density.

Then, it is not difficult to prove, that the dynamical equation for the $n = n_c$ quantity, which follows from Eqs. (2.1.5) and (2.1.7), reads:

$$n_{,\,\tilde{t}_\alpha} = -\mathcal{A}_2^{-1}\dot{\tilde{E}}_\alpha, \tag{2.1.8}$$

where

$$\mathcal{A}_2 = \ddot{\mathcal{S}}_{2\,c} = \ddot{\sigma}_c + \sum_\beta \ddot{\tilde{E}}_{\beta\ c}\tilde{t}_\beta \tag{2.1.9}$$

Here $\dot{\tilde{E}}_\alpha$ is the N-column with the elements $\partial\tilde{E}_\alpha/\partial n_k$, whereas $\ddot{\sigma}_c$ and $\ddot{\tilde{E}}_{\alpha\,c}$ are the $N \times N$ symmetric matrices with the coefficients $\partial^2\sigma/\partial n_k\partial n_l$ and $\partial^2\tilde{E}/\partial n_k\partial n_l$ (calculated at the 'classical trajectory' $n = n_c$), respectively. Our main statement is that this dynamical system of partial differential equations is consistent for the arbitrary functions $\sigma(n)$ and $\tilde{E}_\alpha(n)$ taken as initial data of the solution. This means, that in the n-representation our dynamical problem has not any additional and artificial restrictions – in complete agreement with the announcement given at the beginning of this part of the article. The only restriction which must be put on the system dynamics is the original commutation relation for the Hamiltonians of the theory, in view of the starting general quantum solution explored. Note, that we mean n as the column of the height N here and in the all following relations.

It is really important to stress, that the functions $E_\alpha(n)$, $\sigma(n)$ do not depend on the Planck constant $\hbar$. Thus, in the classical regime our quantum relations have not any 'rule parameter', related to this actually small physical quantity. Furthermore, we need in a new 'rule parameter' to define a correct classical theory derivation. It is clear, that it can be found for the macroscopical system as some proportionality coefficient, which defines a classical scale for its 'big' mass, charges, etc. In the other words, there is not any reason for extracting of the quantum theory with its classical regime using only the limit procedure $\hbar \to 0$. We relate this resonance structure of the kinematics of the theory to the natural parameter ε such that the classical limit for the functions $\bar{E}$, $\bar{\sigma}$, etc., must be proportional to the quantity $1/\varepsilon$.

In the next section, it will be shown that the limit procedure $\varepsilon \to 0$, together with the procedure, defined by the relation $\hbar \to 0$, is enough for the correct formulation of the classical theory. Here it is necessary to note, that the limit at $\hbar \to 0$ is important to calculation of the set of canonical moments (their average values) $p_k = -i\hbar\partial/\partial_{n_k}$ in the theory. The result reads:

$$p_k = \left(\frac{\partial \mathcal{S}_1 \left(\tilde{t}_\alpha, \, q \right)}{\partial q_m} \right)_c , \tag{2.1.10}$$

where we have combined N moment variables to the single N-column quantity. In this formula $\mathcal{S}_1$ means the real part of the phase $\mathcal{S}$ of the wave-function Ψ in the n-representation of the theory, which is defined by the relation

$$\Psi(n, t) = C(n) \exp \left(\frac{i}{\hbar} \mathcal{S} \right). \tag{2.1.11}$$

It is easy to calculate its explicit form in the situation under consideration; the result reads:

$$\mathcal{S}_1(t, \, n) = \lambda(n) + E(n)t - \frac{\hbar^2}{4} \Gamma(n)\beta. \tag{2.1.12}$$

Here we mean again the original theory with the double time parameter, and define the function $\lambda(n)$ according to the relation

$$C(n) = \sqrt{\rho(n)} \exp \left(-\frac{i}{\hbar} \lambda \right). \tag{2.1.13}$$

Then, in the explicit form,

$$p = \dot{\lambda} + \dot{E} t_1, \tag{2.1.14}$$

and for the time derivatives of the moment column p on obtains the equation $p_{\tilde{t}_\alpha} = \dot{E}\delta_{1,\alpha} + \mathcal{A}_1 n_{,\tilde{t}_\alpha}$, where

$$\mathcal{A}_1 = \ddot{\mathcal{S}}_{1\,c} = \ddot{\lambda} + \ddot{E}\, t. \tag{2.1.15}$$

Thus, using the relation (2.1.8), one concludes, that

$$p_{,\tilde{t}_\alpha} = \dot{E}\delta_{1,\alpha} + \mathcal{A}_1\mathcal{A}_2^{-1}\dot{E}_\alpha. \tag{2.1.16}$$

It is clear, that the relations (2.1.8) and (2.1.16) form the pair of Hamiltonian equations for the theory under consideration. The main statement is that this system of equations is consistent, i.e., that the mixed time derivatives for coordinate and moment variables are equal to the inverse one. This means, that this system of classical equations is a correct one.

Our second statement is related to irreversibility of evolution of the system, defined by the Hamiltonian Eqs. (2.1.8)–(2.1.16), combined with some specified temperature regime $\beta = \beta(t)$. We consider below isothermal and adiabatic regimes again, as it had been performed in the quantum part of this work. To prove the presence of arrow of time in these physically important regimes, let us choose a representation, where $\Gamma = \Gamma(n_1)$, and $E = E(n_2)$. It is clear, that this representation exists for the theory taken (i.e., for the theory with commuting Hamiltonian operators). From the Hamiltonian equations it follows, that

$$\begin{aligned} n_{1,t} &= -\Gamma^{'}(\mathcal{A}_2^{-1})_{11}, \qquad & n_{2,t} &= -\Gamma^{'}(\mathcal{A}_2^{-1})_{21}, \\ n_{1,\beta} &= -E^{'}(\mathcal{A}_2^{-1})_{12}, \qquad & n_{2,\beta} &= -E^{'}(\mathcal{A}_2^{-1})_{22}, \end{aligned} \tag{2.1.17}$$

where the prime means the derivative in respect to corresponding single variable of the function under consideration. We take the function Γ as the waiting Lyapunov one, according to the analysis performed in the quantum part of this work. Then, for the total t-derivative of this function, i.e. for the quantity $\Gamma_{;t} = \Gamma^{'} n_{1;t} = n_{1,t} + \beta^{'} n_{1,\beta}$, one can calculate its explicit form using Eqs. (2.1.17). For the isothermal case ($\beta = \text{const}$) the result reads:

$$\Gamma_{;t} = -\left(\Gamma^{'}\right)^2 \left(\mathcal{A}_2^{-1}\right)_{11}. \tag{2.1.18}$$

It is clear, that $\Gamma_{;t} < 0$. In the adiabatic situation with $E = \text{const}$ one obtains

$$\beta^{'} = -\frac{\Gamma^{'}}{E^{'}} \frac{\left(\mathcal{A}_2^{-1}\right)_{21}}{\left(\mathcal{A}_2^{-1}\right)_{22}} \tag{2.1.19}$$

for its temperature regime, and

$$\Gamma_{;t} = -\frac{(\Gamma)^2}{\left(\mathcal{A}_2^{-1}\right)_{22}} \begin{vmatrix} \left(\mathcal{A}_2^{-1}\right)_{11} & \left(\mathcal{A}_2^{-1}\right)_{12} \\ \left(\mathcal{A}_2^{-1}\right)_{21} & \left(\mathcal{A}_2^{-1}\right)_{22} \end{vmatrix} \tag{2.1.20}$$

for the total time dependence of the Hamiltonian function Γ. It is clear, that in the both thermodynamical regimes considered here one deals with decreasing evolution of the function

$\Gamma(t)$ for the arbitrary initial data taken. This proves the Lyapunov nature of this quantity and the presence of arrow of time in the dynamics of the classical system under investigation. Note, that in the both situations considered the inequality $\Gamma_{;t} < 0$ is guaranteed by the positive definition of the matrix $\mathcal{A}_2$, which we have supposed for the solution of the dynamical equations to deal with maximum of the quantity $\mathcal{S}_2$ on the whole 'classical trajectory' $q = q_c(t)$ of the system.

It is necessary to stress, that one has a possibility of taking of any value of the matrix field $\ddot{\sigma}(n)$, which provides the resonance-like character of initial data and the resonance-dominated dynamical history of the probability density $P(t,\, n)$ for the classical object. In fact, this symmetric matrix field modifies the dynamics of this object in the same way as a curved metrics 'acts' on the trajectory of the point mass (or on the bosonic string, in view of the two-dimensional nature of the problem under consideration). The only difference is related to nature of the dynamical variables in these two theories. Actually, here we deal with the 'action-angle variables' of the system, whereas in the standard mechanics and string theory the corresponding quantities are the usual physical coordinates. Also, both our parameters of the dynamics are time-like ones (in the case of $A = 2$), whereas their string theory analogies have the time-like and space-like nature. From formal point of view, our classical dynamics is the most closely related to the dynamics of the bosonic string in the background metric field. In the our case, this 'gravitation' is modeled by the matrix $\ddot{\sigma}(n)$, which describes the 'curvature effects' in the N-dimensional space of the canonical variables n_k.

At the end of this section let us note, that one cannot take the initial value $n_0 = n_{c\,0}$ of the 'classical object position' and the 'phase space metric' $\ddot{\sigma}(n)$ in the independent way. Actually, n_0 denotes the 'maximum point' of the probability density distribution at the 'initial times' $\tilde{t}_\alpha$, whereas the 'metric' $\ddot{\sigma}(n)$ defines this 'metric' completely. Thus, it is obvious an existence of the relation between n_0 and $\ddot{\sigma}(n)$. This means, that in the modified Hamiltonian dynamics under consideration initial data must be consistent with the dynamical characteristics of the system.

2.2 Modified Hamilton equations

Below we use an experience obtained in the study of the modified classical dynamics using the action-angle variables, for investigation of the theory in terms of its original canonical coordinates q and moments p. Our first goal is to derive the corresponding equations of motion of the Hamilton type for this theory and to check their consistence. Realization of this program seems really important in view of the highly non-trivial procedure of search of the action-angle variables n for the system in the general case.

Starting this realization, we define the (hard) classical regime for the quantum theory as the double limit at $\varepsilon \to 0$ and $\hbar \to 0$, which is taken in the appropriate way in the all exact quantum relations. The main facts, which we have established in the study of classical dynamics of the system in terms of the action-angle variables, are collected in the decompositions

$$\mathcal{S} = \mathcal{S}_1 + i\frac{\hbar}{2}\mathcal{S}_2, \quad \mathcal{H} = \tilde{E}_2 - i\frac{\hbar}{2}\tilde{E}_1, \quad \tau = \tilde{t}_1 - i\frac{\hbar}{2}\tilde{t}_2, \tag{2.2.1}$$

where all the quantities written are the classical ones (i.e., they are $\hbar$-free ones; in other words – they are the zero $\hbar$-power magnitudes). We again use the notation $\tilde{E}_1 = \gamma$, $\tilde{E}_2 = E$, $\tilde{t}_1 = t$, $\tilde{t}_2 = \beta$, and parameterize the wave-function Ψ in terms of the complex phase $\mathcal{S}$ as $\Psi = \exp(i\mathcal{S}/\hbar)$. Thus, we plan to derive dynamical equations of the classical regime of the quantum theory under consideration, using the $\hbar$-power structure of the theory constituents established in the study of its n-representation.

Note, that one can restrict this analysis by the study of the single limit $\varepsilon \to 0$ only. In the corresponding quasi-classical theory, one deals with well-localized objects, but the dynamics of theory contains the free parameter $\hbar$. It is really interesting to note, that in this variant of the theory one obtains a dynamical model with highly symmetric algebraic structure of the symplectic type. We develop in details corresponding general formalism in the next sections of this work.

The Schrodinger's equation $i\hbar\Psi_{,\tau} = \mathcal{H}\Psi$ and the analytical condition $\Psi_{,\tau^*} = 0$ leads one to the following relations:

$$\begin{aligned}
\mathcal{S}_{1,\,\tilde{t}_1} + i\frac{\hbar}{2}\mathcal{S}_{2,\,\tilde{t}_1} &= -\left(\hat{\tilde{E}}_2 - i\frac{\hbar}{2}\hat{\tilde{E}}_1\right), \\
\mathcal{S}_{1,\,\tilde{t}_2} + i\frac{\hbar}{2}\mathcal{S}_{2,\,\tilde{t}_2} &= i\frac{\hbar}{2}\left(\hat{\tilde{E}}_2 - i\frac{\hbar}{2}\hat{\tilde{E}}_1\right).
\end{aligned} \tag{2.2.2}$$

Here, the hatted quantities are defined as

$$\hat{\tilde{E}}_\alpha = \Psi^{-1}\tilde{E}_\alpha\Psi, \tag{2.2.3}$$

and mean nontrivial functions of the coordinates q and some set of the derivatives of the complex phase $\mathcal{S}$. The last dependence originates from the $p = -i\hbar\partial/\partial q$-terms in the operators $\tilde{E}_\alpha$.

To 'feel' a nature of the hatted quantities, it is convenient to calculate the simplest polynomial moment functions. In fact, we need in the linear and quadratic ones only for the consistent development of the 'hard' classical limit of the theory. Performing straightforward calculations, it is not difficult to prove, that

$$\hat{p}_k = \mathcal{S}_{,q_k}, \qquad \widehat{p_k p_l} = \mathcal{S}_{,q_k}\mathcal{S}_{,q_l} - i\hbar\mathcal{S}_{,q_k q_l}. \tag{2.2.4}$$

Using these relations, one can prove, that

$$\hat{\tilde{E}}_\alpha = \tilde{E}_\alpha(q,\, \mathcal{S}_{1,\,q}) + i\frac{\hbar}{2}\left[\tilde{E}_{\alpha,\,p^T}\mathcal{S}_{2,\,q} - \mathrm{tr}\left(\tilde{E}_{\alpha,\,pp^T}\mathcal{S}_{1,\,qq^T}\right)\right], \tag{2.2.5}$$

up to $\sim \hbar$ terms. Note, that we explore the compact collective notation, which can be 'decoded' as $\tilde{E}_{\alpha,\,p^T}\mathcal{S}_{2,\,q} = \sum_k \tilde{E}_{\alpha,\,p_k}\mathcal{S}_{2,\,q_k}$ and $\mathrm{tr}\left(\tilde{E}_{\alpha,\,pp^T}\mathcal{S}_{1,\,qq^T}\right) = \sum_{k\,l}\tilde{E}_{\alpha,\,p_k p_l}\mathcal{S}_{1,\,q_k q_l}$, etc., in the corresponding relations.

Then, from Eqs. (2.2.2) it follows, that the relations, which define the action-like quantities $\mathcal{S}_\alpha$ in the regime taken, read:

$$\begin{aligned} \mathcal{S}_{1,\,\tilde{t}_1} &= -\tilde{E}_2(q,\, \mathcal{S}_{1,\,q}), \\ \mathcal{S}_{1,\,\tilde{t}_2} &= 0, \end{aligned} \tag{2.2.6}$$

and

$$\begin{aligned} \mathcal{S}_{2,\,\tilde{t}_1} &= \tilde{E}_1(q,\, \mathcal{S}_{1,\,q}) - \tilde{E}_{2,\,p^T}(q,\, \mathcal{S}_{1,\,q})\mathcal{S}_{2,\,q} + \mathrm{tr}\left[\tilde{E}_{2,\,pp^T}(q,\, \mathcal{S}_{1,\,q})\ \mathcal{S}_{1,\,qq^T}\right], \\ \mathcal{S}_{2,\,\tilde{t}_2} &= \tilde{E}_2(q,\, \mathcal{S}_{1,\,q}). \end{aligned} \tag{2.2.7}$$

It is seen, that the demanding of $\hbar \to 0$ leads to the absolutely non-symmetric structure of the dynamical equations. In these equations, roles of the operators E and Γ are non-identic transparently. This fact follows from the absolutely asymmetric $\hbar$-dependence of the right sides of the relations (2.2.2).

Let us discuss the general physical sense of the system (2.2.6)-(2.2.7). First of all, from the second equation of (2.2.6) it follows that $\mathcal{S}_1$ does not depend on the inverse temperature β, i.e., $\mathcal{S}_1 = \mathcal{S}_1(t,\, q)$. Then, the first equation (2.2.6) is the standard Hamilton-Jacoby equation exactly (because $\tilde{E}_2$ is the energy dynamical variable – the standard Hamiltonian of the system). This means, that $\mathcal{S}_1$ is a classical action $\mathcal{S}_a = \mathcal{S}_a(t,\, q)$ of the system, which satisfies the Hamilton-Jacoby equation

$$\mathcal{S}_{a,\,t} = -E\left(q,\, \mathcal{S}_{a,\,q}\right). \tag{2.2.8}$$

Then, the physical sense of the function $\mathcal{S}_2$ can be obtained using the study of the system (2.2.7). Actually, from the second equation of (2.2.7) it follows, that

$$\mathcal{S}_2 = \mathcal{S}_e + E\beta, \tag{2.2.9}$$

where the function $\mathcal{S}_e = \mathcal{S}_e(t,\, q)$ does not depend on β. This new 'action-like function' satisfy the equation

$$\mathcal{S}_{e,\,t} = \gamma(q,\, \mathcal{S}_{a,\,q}) + \mathrm{tr}\left[\tilde{E}_{2,\,pp^T}(q,\, \mathcal{S}_{a,\,q})\; \mathcal{S}_{a,\,qq^T}\right], \tag{2.2.10}$$

as it can be easily verified, using the first equation of (2.2.7) and the definition of $\mathcal{S}_e$ given by Eq. (2.2.9).

Our statement is: the quantity $\mathcal{S}_e$, being calculated on the 'classical trajectory' $q = q_c$, coincides with the classical entropy of the system. This means, that the function $S_e(t,\, q_c(t, \beta))$ is the standard classical entropy, which corresponds to the system dynamics in the conventional thermodynamic framework. To prove this 'shocking statement', one must take into account, that the classical entropy can be calculated using the standard relation

$$\mathcal{S}_e = -\ln \mathcal{Z} - \beta\, E, \tag{2.2.11}$$

and that in the classical limit $\varepsilon \to 0$ one obtains

$$\mathcal{Z} = \exp\left(-\mathcal{S}_{2\,c}\right). \tag{2.2.12}$$

Finally, one concludes, that the modified 'hard' classical dynamics is completely described in terms if the classical action and entropy function of the theory. The incorporation of the entropy into the general Hamiltonian formalism allows one to predict an arising of the arrow of time in the corresponding dynamics.

Namely, the main statement, related to the system (2.2.6)-(2.2.7), is that this system is actually consistent. This property is not obvious, because the system was derived by the help of the perturbation expansion in respect to the Plank constant. Then, 'cut' of the all $\hbar$-dependent terms performed during a derivation of the system (2.2.6)-(2.2.7) could destroy the consistency of the original quantum equations. Thus, one must compare an equivalence of the mixing time derivatives to be sure in correctness of the all dynamical results established during the study of this theory.

Proof of the consistence under discussion can be performed using the corresponding straightforward calculations. First of all, the identity

$$\mathcal{S}_{1,\,\tilde{t}_1\tilde{t}_2} = \mathcal{S}_{1,\,\tilde{t}_2\tilde{t}_1} \tag{2.2.13}$$

is evident in view of the form of the second equation from the subsystem (2.2.6) (it is easy to see, that both these mixed derivatives are equal to zero). Then, in the case of the equations (2.2.7) one has not a subsystem, because the defining $\mathcal{S}_2$ relations include the quantities related to $\mathcal{S}_1$ in their right sides. However, using the total system (2.2.6)-(2.2.7), one concludes, that

$$\mathcal{S}_{2,\,\tilde{t}_1\tilde{t}_2} = \mathcal{S}_{2,\,\tilde{t}_2\tilde{t}_1} = -\tilde{E}_{2,\,p^T}\tilde{E}_{2;\,q}. \tag{2.2.14}$$

This means, that both the action-like quantities $\mathcal{S}_\alpha$ are defined by the pure *classical* system of the equations correctly, i.e. in the consistent form. This means, that the classical theory under construction *exists*, because it is defined completely by the equations (2.2.6)-(2.2.7). Actually, we relate this classical theory with some special solutions of this consistent system of partial differential equations – with such solutions, which allow the highly resonance character in terms of the function $\mathcal{S}_2$. Taking the resonance structure of this type as initial data, one can study its consistent dynamics using corresponding conclusions of Eqs. (2.2.6)-(2.2.7). These conclusions must have sense of the modified Hamilton equations. Now we start their derivation.

This derivation is based on the use of the same approach, as the one used for the study of classical dynamics of the system under consideration in terms of its action-angle variables. Namely, we start with differentiation of the extremum relation

$$\left(\mathcal{S}_{2,\,q_k}\right)_c = 0 \tag{2.2.15}$$

in respect to the time variables $\tilde{t}_\alpha$. Here $(\mathcal{S}_{2,\,q_k})_c = \mathcal{S}_{2,\,q_k}(t,\,q(t))$, where $q(t)$ means the classical 'trajectory', which minimizes the $\mathcal{S}_2(t,q)$ value. Starting from this point, let us come back to the physical notation t,β and E,Γ in the all relations. Performing the calculations, one concludes, that

$$\begin{aligned} q_{,\,t} &= E_{,p} - \mathcal{A}_2^{-1}\Gamma_{*;q}, \\ q_{,\,\beta} &= -\mathcal{A}_2^{-1}E_{;q}, \end{aligned} \tag{2.2.16}$$

where the index '$;q$' is used to denote the total q-derivative taken, and we have introduced the 'renormalized' decay function Γ_* as:

$$\Gamma_* = \Gamma + \mathrm{tr}\left[\tilde{E}_{2,\,pp^T}(q,\,\mathcal{S}_{1,\,q})\;\mathcal{S}_{1,\,qq^T}\right]. \tag{2.2.17}$$

Then, for the moments $p = \mathcal{S}_{1,\,q}$, the corresponding defining relations read:

$$\begin{aligned} p_{,\,t} &= -E_{,\,q} - \mathcal{A}_1\mathcal{A}_2^{-1}\Gamma_{*;q}, \\ p_{,\,\beta} &= -\mathcal{A}_1\mathcal{A}_2^{-1}E_{;q}. \end{aligned} \tag{2.2.18}$$

Note, that the t-dependence of the canonical coordinates and moments includes the traditional 'pure Hamiltonian' part. However, the complete right sides of the motion equations are modified by the presence of dispersion-like structures hidden in the quantities $\mathcal{A}_\alpha$. Eqs. (2.2.16), (2.2.17) describe, together with the system (2.2.6)–(2.2.7), all the 'hard' classical dynamics of the system under consideration. This dynamics can be named as 'modified Hamiltonian dynamics'; our main goal is to study its irreversibility properties.

However, the initial point in the study of this system of partial differential equations must concern its consistence. First of all, it is not difficult to prove, that the moment relations (2.2.18) are consistent if the ones (2.2.16) for the canonical coordinates allow this property. To establish this fact, it is convenient to take into account, that these systems are related as

$$p_{,t} = -E_{;q} + \mathcal{A}_1 q_{,t}, \quad p_{,\beta} = -\mathcal{A}_1 q_{,\beta}. \tag{2.2.19}$$

The proof of the equivalence $q_{,t\beta} = q_{,\beta t}$ is more difficult. Nevertheless, this consistence condition is actually take place. Using straightforward calculations, it is possible to show, that

$$q_{,t\beta} - q_{,\beta t} = \left(\mathcal{S}_{2,t\beta} - \mathcal{S}_{2,\beta t}\right)_{;q}, \tag{2.2.20}$$

so the consistency of the Hamiltonian equations follows from the consistency for the phase quantities $\mathcal{S}_\alpha$. Thus, our double limit $\varepsilon \to 0$, $\hbar \to 0$ taken in the consistent original quantum dynamics leads to the correct classical regime of the system under consideration. We name it as 'modified Hamiltonian dynamics' in view of the obvious reasons.

Now let us consider the problem of presence of the arrow of time in the modified Hamiltonian dynamics formulated above. First of all, the total t- and β- derivatives, calculated for the energy and decay operators E and Γ, read:

$$\begin{array}{ll} E_{,t} = -\left(E,\, E\right), & E_{,\beta} = -\left(E,\, E\right), \\ \Gamma_{,t} = -\left(\Gamma,\, \Gamma\right) + \left\{E,\, \Gamma\right\}, & \Gamma_{,\beta} = -\left(\Gamma,\, E\right). \end{array} \tag{2.2.21}$$

Here, for example, the scalar product of the quantities Γ and E is denoted as $(\Gamma,\, E)$, and defined according to the relation

$$(\Gamma,\, E) = \Gamma_{;q^T} \mathcal{A}_2^{-1} E_{;q}. \tag{2.2.22}$$

Then, the term $\{E,\, \Gamma\}$ means the standard classical Poisson brackets. Below we suppose, that (as in the quantum theory case) the commutation relation

$$\{E,\, \Gamma\} = 0 \tag{2.2.23}$$

takes place for the theory under consideration. Note, that one must take into account the *total* dependence on q in the calculation of the derivative ; q of the corresponding dynamical quantity. Also, it is important to remember, that the matrix $\mathcal{A}_2$ is positively defined, so the scalar production of the any (nontrivial) dynamical quantity to itself is the positive magnitude always.

For the total t-dependence, i.e., in the case of the temperature curve $\beta = \beta(t)$ defined, one deals with the decay dynamical variable, which satisfies the following relation:

$$\Gamma_{;\,t} = \Gamma_{,\,t} + \beta^{'}\Gamma_{,\,\beta}. \tag{2.2.24}$$

For the isothermal regime of evolution of the system (where $\beta(t) = \mathrm{const}$) one obtains, that

$$\Gamma_{;\,t} = -\left(\Gamma,\,\Gamma\right) < 0. \qquad \text{(isothermal case)} \tag{2.2.25}$$

Thus, in the isothermal regime the decay variable Γ decreases for any initial data taken. This means the presence of the arrow of time in the corresponding dynamics of the system. Then, in the adiabatic case (where $E_{;\,t} = E_{,\,t} + \beta^{'}E_{,\,\beta} \equiv 0$), one obtains, that

$$\beta^{'} = -\frac{(E,\,\Gamma)}{(E,\,E)}. \tag{2.2.26}$$

In derivation of this and previous results we have used Eq. (2.2.21). After the substitution of Eq. (2.2.26) into Eq. (2.2.24) and additional use of Eq. (2.2.21), we finally obtain, that

$$\Gamma_{;\,t} = -\frac{(\Gamma,\,\Gamma)}{(E,\,E)}\left(1 - \frac{(E,\,\Gamma)^2}{(E,\,E)\,(\Gamma,\,\Gamma)}\right). \tag{2.2.27}$$

Note, that in the adiabatic case one has again the evolution with

$$\Gamma_{;\,t} < 0, \qquad \text{(adiabatic case)} \tag{2.2.28}$$

as it follows from Eq. (2.2.27) and from the inequality of Cauchy-Bounyakowsky. Thus, in the adiabatic case we deal also with the time-irreversible classical dynamics.

Note, that all the results of this section exactly correspond to the ones obtained in the quantum part of this article. Namely, the only concretization is related to explicit realization of the scalar products used in the calculations. Also we would like to stress, that Eq. (2.2.17) defines the classical value of the decay variable Γ correctly. This follows from the 'expected form' of the relations (2.2.21)–(2.2.28), presented above, which are written down in terms of this quantity. Thus, in the classical limit the decay dynamical variable obtains the 'quantum shift' expressed in the form of trace term presented above. This term does not destroy the irreversible structure of the theory under consideration.

2.3 Uncertainty relations in classical theory

In this section, we clarify a sense of the quantities $\mathcal{A}_\alpha$ which had arisen during the study of the classical dynamics of the system. Our goal is to show, that these variables are closely related to dispersion characteristics of the theory. The main statement of this section is that the 'soft' classical regime possesses famous uncertainty relations which are well known in the quantum theory framework. It is shown, that this remarkable fact takes place on the kinematic level of the classical theory. The 'soft character' of the statement is really natural, because in the 'hard' regime with $\hbar = 0$ one obtains the trivial classical identity from the original quantum inequalities. However, the simple dispersion sense of the quantities $\mathcal{A}_\alpha$ conserves also in the 'hard' variant of the classical theory too.

First of all we would like to stress, that the average value $\bar{f}$ of the quantum variable $f = f(t_\alpha, q)$ we calculate using the standard relation

$$\bar{f} = \int dq \rho f, \tag{2.3.1}$$

where $\rho = w/\mathcal{Z}$. In the general operator case with $f = f(t_\alpha, q, p)$, where p is the column of the height N, constructed from the standard canonical moment operators, we introduce the convenient quantity $\hat{f} = \Psi^{-1} f \Psi$. Then, for the average value $\bar{f}$ of this general quantum variable, one can use the relation (2.3.1), modified by the substitution $f \to \hat{f}$ into its integral term.

For the case of 'soft' classical regime we suppose only, that the theory describes some point-like (or corpuscular) object. This means, that the probability density has the 'resonance-like' maximum on the classical trajectory of the system. Our goal is to calculate all the fundamental dispersion characteristics (i.e., all the correlations for the total set of the canonical variables) for this system. It is clear, that these correlations describe a behavior of the point-like object under consideration at the nearest vicinity of its classical trajectory in the phase space.

We extract from the solution taken the small parameter ε, which defines the resonance character of the probability density of this solution. Then, in the limit of $\varepsilon \to 0$ we lead to the 'soft' variant of the classical theory under discussion. Performing this limit procedure, for the scale factor $\mathcal{Z}$, one obtains the following result:

$$\mathcal{Z} = e^{-2S_{2\,c}} \sqrt{\frac{(\pi)^N}{\det \mathcal{A}_2}} \left(1 + o\left(\varepsilon\right)\right), \tag{2.3.2}$$

where the well-known Laplace integral theorem had been applied, and the notations defined in the previous section had been used. Below we neglect all the terms $o(\varepsilon)$ in respect to the leading asymptotic in the all relations. However, for calculation of correlations of the canonical coordinates and moments, it is important to save in the asymptotic decompositions all the terms up to ε. In view of this reason, the main work relation for the average quantity $\bar{f}$ reads:

$$\bar{f} = \hat{f}_c + \frac{1}{4}\mathrm{tr}\left(\ddot{\hat{f}}_c \mathcal{A}_2^{-1}\right). \tag{2.3.3}$$

Here $\ddot{\hat{f}}$ is the matrix with the coefficients $\ddot{\hat{f}}_{c\,mn} = (\hat{f}_{;q_m q_n})_c$. Note, that the index '$_{;q_m}$' means again the total derivative of the corresponding dynamical variable in respect to the canonical coordinate q_m.

It is important to stress, that in Eq. (2.3.3), one takes into account all terms up to quadratic ones for the both functions $\mathcal{S}_2$ and $\hat{f}$, decomposed to the Tailor series in the surrounding of the 'extremal point' $q = q_c$. Actually, all the higher derivative terms form $o(\varepsilon)$ modes; we neglect them as it was written above. In fact, here we restrict our study by consideration of the multidimensional 'Gaussian bell' around the 'classical trajectory' of the system. Namely, we perform the representation

$$\mathcal{S}_2 = \mathcal{S}_{2\,c} + \frac{1}{2}\left(q - q_c\right)^T \mathcal{A}_2 \left(q - q_c\right) + \ldots, \tag{2.3.4}$$

where it was used the extremal condition $(\dot{\mathcal{S}}_2)_c = 0$. Thus, the exponential kernel in the integral form, which defines the average values, reads:

$$e^{-\mathcal{S}_2} \cong \exp\left[-S_{2\,c} - \frac{1}{2}\left(q - q_c\right)^T \mathcal{A}_2 \left(q - q_c\right)\right], \tag{2.3.5}$$

where $\mathcal{A}_2$ is the positively defined matrix. This last relation, being approximated by its leading term at the right side, defines the N-dimensional Gaussian bell. In this connection, the correlation parameters of the system are closely related to the 'bell's parameters'. In particular, we suppose, that the matrix $\mathcal{A}_2$ and the 'resonance parameter' ε are related as $\mathcal{A}_2 \sim 1/\varepsilon$, so in the limit case of $\varepsilon \to 0$, $\varepsilon \neq 0$, one obtains the asymptotic results (2.3.2) and (2.3.3). Note, that our 'hard' variant of the modified classical dynamics based on the use of the terms up to the terms $\sim \varepsilon$, see Eq. (2.3.3).

To develop the general formalism, which describes the classical limit of the dispersion characteristics of the theory, let us define the correlation $f \circ g$ of the quantum variables f and g according to the relation

$$f \circ g = \overline{fg + gf} - 2\bar{f}\bar{g}, \tag{2.3.6}$$

where one must calculate the right side up to the $\sim \varepsilon$ accuracy. These quantities, being introduced to the theory, describe non-equivalence of the average value of the product of the dynamical variables and the product of the average values of them. It is easy to see, that the squared dispersion $\mathcal{D}_f^2$ of the dynamical variable f is related to the correlation quantity $f \circ f$ as

$$\mathcal{D}_f^2 = \frac{1}{2} f \circ f. \tag{2.3.7}$$

Thus, a calculation of the correlations is equivalent to the calculation of the dispersion-like characteristics of the system. Our goal is to derive the classical limit of the quantum correlations for the canonical coordinates and moments.

To perform this work, we use the Laplace integral decomposition formula (2.3.3). The result read:

$$\begin{aligned} (q_m \circ q_n)_c &= \left(\mathcal{A}_2^{-1}\right)_{mn}, \\ (q_m \circ p_n)_c &= \hbar \left(\mathcal{A}_2^{-1}\mathcal{A}_1\right)_{mn}, \\ (p_m \circ p_n)_c &= \hbar^2 \left(\mathcal{A}_2 + \mathcal{A}_1\mathcal{A}_2^{-1}\mathcal{A}_1\right)_{mn}. \end{aligned} \tag{2.3.8}$$

Thus, we have proved the statement formulated at the beginning of this section: the quantities $\mathcal{A}_\alpha$ actually define all fundamental correlations in the classical theory under consideration. These quantities, as it is seen from Eqs. (2.2.16), (2.2.18), enter to the right sides of the Hamiltonian equations. Thus, the 'hard' variant of classical dynamics under consideration takes into account classical limit of the dispersion characteristics of the system.

It is important to note, that one can use an alternative momentum representation of the original quantum theory. This representation is the most natural for constructing and study of the classical wave limit of this theory. It is easy to understand, that one must incorporate all dispersion characteristics to consider both the corpuscular and wave classical regimes in framework of the single consistent theoretical scheme. In doing so, one must be sure, that uncertainty relations are 'about minimized' during the total dynamical history of the system. In the opposite case, the wave pocket can lose its local nature, so the classical theory description become destroyed without any kinematical 'control' under this significant process. Thus, the dispersion characteristics, or correlations, play extremely important role in the physically well-motivated formulation of the classical theory originated from the fundamental quantum one.

Then, using the relation between the dispersions and correlations, it is easy to check, that the uncertainty relation

$$\mathcal{D}_{q_m}\mathcal{D}_{p_m} \geq \frac{\hbar^2}{4} \tag{2.3.9}$$

is equivalent to the inequality

$$(q_m \circ q_n)\ (p_m \circ p_n) \geq \hbar^2 \tag{2.3.10}$$

written in terms of the correlations. Our statement is that the inequality (2.3.9) remains conserved under the classical limit procedure defined above. Actually, using induction in respect to N, it is not difficult to prove, that

$$\left(\mathcal{A}_2^{-1}\right)_{mm}\left(\mathcal{A}_2 + \mathcal{A}_1\mathcal{A}_2^{-1}\mathcal{A}_1\right)_{mm} \geq 1 \tag{2.3.11}$$

for any matrix $\mathcal{A}_1$ and for any positively-defined matrix $\mathcal{A}_2$. This means, that the uncertainty relations can be incorporated correctly into the structure of the 'soft' classical theory on the pure kinematical level.

Another important quantum inequality between the dispersion characteristics of the system fixes a signature of the 'quasi-metric', which can be related to the total set of the fundamental correlations. It is given by the relation

$$(q_m \circ q_m)\ (p_m \circ p_m) \geq (q_m \circ p_m)^2\,, \tag{2.3.12}$$

also remains true in the 'soft' classical limit of the theory under consideration. This quantum relation, which has the explicit Cauchy-Bouniakowsky type, transforms into the classical inequality

$$\left(\mathcal{A}_2^{-1}\right)_{mm}\left(\mathcal{A}_2 + \mathcal{A}_1\mathcal{A}_2^{-1}\mathcal{A}_1\right)_{mm} \geq \left(\left(\mathcal{A}_2^{-1}\mathcal{A}_1\right)_{mm}\right)^2, \tag{2.3.13}$$

which is actually true for the arbitrary matrices with the properties supposed. Both the classical inequalities presented above are important for the study of the kinematics of the theory under construction. They clarify the features of the dispersion characteristics of motion of the classical object in the phase space of the system. To demonstrate a significance of the correlations in the kinematical structure of the classical system, one needs in development of some group formalism related to the correlations. We establish it in the next section.

2.4 Symplectic formalism and canonical maps

First of all, let us unify all the fundamental correlations of the theory to the following $2N \times 2N$ 'correlation matrix' $\mathcal{M}$:

$$\mathcal{M} = \begin{pmatrix} q \circ q^T & \hbar^{-1} q \circ p^T \\ \hbar^{-1} p \circ q^T & \hbar^{-2} p \circ p^T \end{pmatrix}. \tag{2.4.1}$$

In the 'soft' classical limit, the explicit form of this block matrix reads:

$$\mathcal{M}_c = \begin{pmatrix} \mathcal{A}_2^{-1} & \mathcal{A}_2^{-1}\mathcal{A}_1 \\ \mathcal{A}_1\mathcal{A}_2^{-1} & \mathcal{A}_2 + \mathcal{A}_1\mathcal{A}_2^{-1}\mathcal{A}_1 \end{pmatrix}, \tag{2.4.2}$$

see Eq. (2.3.8). It is clear, that $\mathcal{M}_c$ is symmetric, i.e.,

$$\mathcal{M}_c^T = \mathcal{M}_c. \tag{2.4.3}$$

The less evident algebraic property of the correlation matrix can be expressed in terms of the quadratic relation

$$\mathcal{M}_c \mathcal{L} \mathcal{M}_c = \mathcal{L}, \tag{2.4.4}$$

where

$$\mathcal{L} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.4.5}$$

This group relation can be easily verified using straightforward calculations and taking into account of Eq. (2.4.2). The relations (2.4.3) and (2.4.4) mean, that the matrix $\mathcal{M}_c$ parameterizes the coset $Sp(2N,R)/U(N)$; in fact it is the canonical null-curvature matrix of the coset written. Also, it is important to note, that the transformation

$$\mathcal{M}_c \to \mathcal{C}\mathcal{M}_c\mathcal{C}^T \tag{2.4.6}$$

preserves both the defining coset relations (2.4.3) and (2.4.4), if

$$\mathcal{C}\mathcal{L}\mathcal{C}^T = \mathcal{L}. \tag{2.4.7}$$

Thus, this transformation does not change algebraic structure (2.4.2) of the dispersion matrix. This transformation seems like a symmetry map in the theory of nonlinear sigma-models, and clarification of its status in the theory under consideration can generate a real progress in its formulation and in the following study.

To realize the corresponding program, let us note, that the relation

$$\mathcal{M} = \mathcal{X} \circ \mathcal{X}^T, \tag{2.4.8}$$

where

$$\mathcal{X} = \begin{pmatrix} q \\ p \end{pmatrix}, \tag{2.4.9}$$

takes place, as it becomes clear immediately in view of the form of the quantum correlation matrix given by Eq. (2.4.1). Thus, in the case of the q-independent transformation matrix $\mathcal{C}$, one deals with the map

$$\mathcal{X} \to \mathcal{C}\mathcal{X} \tag{2.4.10}$$

of the column of the canonical variables. We stand, that this map, where $\mathcal{C} = \mathcal{C}(t_\alpha) \in Sp(2N,\, R)$, is, in fact, the canonical transformation in conventional mechanical sense (see [17]–[18] for close analogies in superstring gravity models).

Actually, it is easy to prove, that

$$\left\{\mathcal{X},\, \mathcal{X}^T\right\} = \mathcal{L}, \tag{2.4.11}$$

where it is taken into account, that the Poisson bracket between the moment p_m and coordinate q_n is $\{p_m,\, q_n\} = \delta_{mn}$. Thus, the group property (2.4.7) for the transformation $\mathcal{C}$ is equal to the usual conservation of the fundamental Poisson brackets. This proves our statement: the $\mathcal{C}$-map is actually the conventional canonical transformation.

Then, the symplectic quantities established allow one to write down the 'soft' Hamilton equations for the system under consideration in their explicitly symplectic invariant form. Here we mean generalization of the special Hamilton equations, written for the 'hard' regime with $\hbar = 0$. Namely, the statement is: in the 'soft' case under discussion with $\tilde{\mathcal{H}}_\alpha = \tilde{\mathcal{H}}_\alpha(q,\, p) = \tilde{\mathcal{H}}_\alpha(\mathcal{X})$, the leading symplectic term (which does not include $\hbar$) in the motion equation of the theory reads:

$$\mathcal{X}_{,t_\alpha} \cong -\mathcal{L}\nabla_{\mathcal{X}}\tilde{\mathcal{H}}_\alpha + \epsilon_{\alpha\beta}\mathcal{M}_c\nabla_{\mathcal{X}}\tilde{\mathcal{H}}_\beta, \tag{2.4.12}$$

where $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha},\ \epsilon_{12} = 1$ is the Levi-Chivita symbol. Here $\tau = t_1 - it_2$ and $\mathcal{H} = \mathcal{H}_1 + i\mathcal{H}_2$ now (let us remember, that $\hbar \neq 0$ in the 'soft' classical regime under consideration). Then, in (2.4.12) the total derivative in respect to $\mathcal{X}$ is understood as the $2N$-column. Also we mean, that the quantities $\tilde{\mathcal{H}}_\alpha$ are considered as the functions calculated in the classical limit given by substitution of $\nabla_q \mathcal{S}_1$ to the p-term into the Hamiltonians. The exact form of the right side of Eq. (2.4.12) contains terms depending on the Plank constant. It is interesting to note that the underlying symplectic structure is totaly $\hbar$-free. Note, that one has a reason to consider this system of equations in the case with finite values of t_2 and $\mathcal{H}_2$, i.e., when $\hbar\beta$ and $\hbar\gamma$ are non-vanishing quantities. Thus, the corresponding 'soft' limit must be 'viewed' at high temperatures and very fast decay processes in the system.

Also, one must answer the natural question, when it is possible to neglect the $\hbar$-depending terms in the modified Hamiltonian equation (2.4.12). It is not difficult to prove, that in the corresponding case the consistency condition $\mathcal{X}_{,t_1t_2} = \mathcal{X}_{,t_2t_1}$ is satisfied if

$$\mathcal{M}_{,t_\alpha} = \mathcal{M}I_\alpha\mathcal{L} - \mathcal{L}I_\alpha\mathcal{M} + \epsilon_{\alpha\beta}\left(\mathcal{L}I_\beta\mathcal{L} + \mathcal{M}I_\beta\mathcal{M}\right), \tag{2.4.13}$$

where $I_{alpha} = \nabla_{\mathcal{X}}\nabla_{\mathcal{X}}^T\mathcal{H}_\alpha$. Finally, Eq. (2.4.13) is consistent itself, if the Hamiltonians $\mathcal{H}_\alpha$ are some linear or quadratic functions in respect to the canonical variables $\mathcal{X}$. Note, that these variables can be related to the original physical coordinates and moments in a highly nontrivial form, so the last restriction is not too hard, as it can be naively understood. For example, one can use the action-angle variables, which can be taken in the corresponding form for any Hamiltonian system of the type under consideration (i.e., one can reach linear or quadratic form of the Hamiltonian functions in the general case).

Then, using straightforward calculations, it is not difficult to prove, that in the isothermal and adiabatical regimes of the thermodynamical evolution, this new ('soft') classical system demonstrates the effect of the irreversibility in its evolution. The proof is very similar to the one performed in the previous section; we will leave it as the not really difficult exercise. It is important to stress, that this leading term in the modified Hamilton equations (2.4.12) is actually invariant under the transformation (2.4.1), (2.4.10).

For study and application of the canonical map established above, let us introduce the following complex symmetric $N \times N$-matrix field $\mathcal{A}$:

$$\mathcal{A} = \mathcal{A}_1 + i\mathcal{A}_2. \tag{2.4.14}$$

Our statement is that the transformation (2.4.6) can be decomposed into set of the maps

$$\mathcal{A} \to \mathcal{A} + \Lambda_1, \quad \mathcal{A} \to \Lambda_2^T\mathcal{A}\Lambda_2, \quad \mathcal{A}^{-1} \to \mathcal{A}^{-1} + \Lambda_3, \tag{2.4.15}$$

where Λ_1, Λ_2 and Λ_3 are the real t_α-dependent matrices, $\Lambda_1^T = \Lambda_1,\ \Lambda_3^T = \Lambda_3$, and $\det\Lambda_2 \neq 0$. It is interesting to note, that these transformations coincide exactly with the matrix-valued hidden symmetries of the General Relativity written in terms of the Ernst (matrix) potential $\mathcal{A}$. Namely, the first map from this set is the shift transformation, the second one is the scale symmetry, whereas the third map is the Ehlers non-linear transformation. A remarkable fact related to the presented decomposition is based on the following trivial statement: the discrete map

$$\mathcal{A} \to \mathcal{A}^{-1} \tag{2.4.16}$$

transforms the shift symmetry into the Ehlers map, and vice-versa. Also, the scale transformation remains itself under this discrete map. One can prove, that the map (2.4.16) corresponds to the transformation (2.4.6) with $\mathcal{C} = \mathcal{L}$, which generates the canonical interchange

$$q \to -p, \qquad q \to p. \tag{2.4.17}$$

Thus, this discrete map is equivalent to the change of representation in the original quantum theory. This means, that kinematics of the classical theory theory under consideration allows the corresponding degree of freedom – the really promising and important fact. From this it follows, for example, that our formalism can be successfully used for description of the both corpuscular and wave limits of the original quantum theory.

Also it is clear, that change of the representation will be reflected on the explicit form of the dispersion characteristics of the system. These dispersion characteristics, or the classical correlations, define effectively some 'tube of motion' in the nearest vicinity of the classical trajectory of the system. They give a key to the kinematic control under the type of the motion: namely, a continuous increasing of the 'tube diameter' makes the average trajectory of the classical object more and more 'quantum-like' one. Also, it seems an obvious fact, that the matrix $\mathcal{M}_c$ can be considered as new field, which makes the phase space of the theory curved in a quasi-gravitational sense. This matrix defines the analogy of the metric in the phase space, which 'interacts' with the 'matter degrees of freedom', collected in the canonical variable $\mathcal{X}$.

At the end of this section let us stress, that the transformation set (2.4.15) can be used for the really dramatic simplification of the dynamical problem for this classical system. The corresponding details for the arbitrary symplectic coset one can find in the literature. The statement is that one can use these transformations to move the $\mathcal{A}_\alpha$-values to the their simplest possible form $\mathcal{A}_1 = 0$, $\mathcal{A}_2 = 1$. Note, that this simplification can be performed during the whole dynamical history of the system (in the any given point of its classical trajectory). One can use this nontrivial fact to perform the dynamical analysis of the complete classical theory in the most simple form.

2.5 Conclusion

In this part we have developed both the kinematical and dynamical parts of the classical theory, which arises in the consistent limits of the quantum theory with arrow of time. we have studied 'hard' and 'soft' classical regimes of the theory, which were defined as the ones describing resonances of the probability density, with $\hbar = 0$ and $\hbar \neq 0$, respectively. We have shown, that it is natural to add all the dispersion characteristics (or the correlations) to the 'usual' set of the fundamental dynamical variables – i.e., to the canonical coordinates and moments of the classical theory. In particular, it is established, that in the 'soft' regime the resulting classical theory possesses a symplectic group of the conventional canonical transformations. It is proved the fulfilment of the uncertainty relations in the presented classical kinematics. We have proposed the quasi-gravitational interpretation of the dispersion parameters and conception of the curved phase space.

We have shown, that the modified classical dynamics allows a well-defined arrow of time (at least for the isothermal and adiabatic regimes). In the 'hard' classical regime, we have studied in details evolution of the classical limit of the original quantum system using the most convenient representation, which corresponds to the action-angle one in the standard classical mechanics. We have derived the modified Hamiltonian equations and prove their consistency condition in the case of the general canonical representation taken.

In the 'soft' representation case, we have established the compact matrix form of the modified Hamiltonian equations which is explicitly invariant under the action of the hidden subgroup of canonical symmetries. We have studied and classified all these symmetries and discover their quasi-Einsteinium structure. In particular, we have extracted from this symmetry subgroup the nonlinear sector of Ehlers transformations. Its arising (as well as the metric-like behavior of the classical correlations in the theory) demonstrates real possibilities for the following development of this theory in the General Relativity direction. It seems natural to wait different applications of the formalism developed in all gravity involved theories – string theory, cosmology, black hole physics [19]–[23]. We hope to present corresponding results in the forthcoming works.

Acknowledgements

We would like to thank prof. B.S. Ishkhanov for his continuous help and attention to us and this work during all time of its preparation. In fact, many of general and special problems considered and solved in this publication were inspired by our numerous discussions and private talks. We think, that they were obtained in many respects owing to this remarkable

human and scientific collaboration.

References

[1] A. Magnon, Arrow of time and reality: in search of a conciliation, Singapore, Singapore: World Scientific (1997).

[2] L. Diosi, Lect. Notes Phys. **633** (2004) 125.

[3] M. Castagnino, L. Lara, O. Lombardi, Int. J. Theor. Phys. **42** (2003) 2487.

[4] I. Prigogine, The arrow of time, Published in *Rome/Pescara (1999), The chaotic universe* 1.

[5] H.B. Callen, Thermodynamics and an introduction to thermodynamics, 2nd edition, John Wiley and Sons, New York (1985).

[6] Sudbery, E., Quantum mechanics and particles of Nature, Cambridge Univ. Press, (1986).

[7] Goldstein, H. Classical mechanics. AddisonWesley Series in Physics, (1980).

[8] I. Prigogine, From Being to Becoming: Time and Complexity in the Physical Sciences, W. H. Freeman and Co., San Francisco (1980).

[9] A. Mejias, L. Di G. Sigalotti, E. Sira, Chaos Solitons Fractals **19** (2003) 773.

[10] H. Aoyama, Nucl. Phys. **B446** (1995) 315.

[11] H. Aoyama, T. Harano, Mod. Phys. Lett. **A10** (1995) 1135. (1995).

[12] S. Biswas, B. Modak, A. Shaw, Gen.Rel.Grav. **32** (2000) 53.

[13] R. Okamoto, S. Fujii, K. Suzuki, Int. J. Mod. Phys. **E14** (2005) 21.

[14] D.B. Fairlie, J. Nuyts, J. Phys. **A38** (2005) 3611.

[15] J. Beckers, J. F., N. Debergh, Giuseppe Marmo, Mod. Phys. Lett. **A16** (2001) 91.

[16] C. M. Bender, S. Boettcher, Phys. Rev. Lett. **80** (1998) 5243.

[17] O. V. Kechkin, Phys. Part. Nucl. **35** (2004) 383.

[18] O. V. Kechkin, Phys. Rev. **D65** (2002) 066006.

[19] A. Linde, Particle physics and inflationary cosmology, Chur, Switzerland: Harwood (1990) Contemporary concepts in physics.

[20] S. W. Hawking, Phys. Rev. **32** (1985) 2489.

[21] V. P. Frolov, I. D. Novikov, Black hole physics: basic concepts and new developments, Kluwer Academic (1998).

[22] R. Brout, S. Massar, R. Parentani, Phys. Rept. **260** (1995) 329.

[23] J. R. David, G. Mandal, S. R. Wadia, Phys. Rept. **369** (2002) 549.